**MATHEMATICS 391**          **FINAL EXAM**          **Fall, 2011**

Part I Answer all questions in this part.

(1) Compute the general solution of each of the following (7 points each):

(a)    $t \dfrac{dy}{dt} + 4y^2 \;\; = 0 \quad (t > 0)$.

(b)    $2xy \, dy \;\; = \;\; (x^2 + y^2) \, dx$.

(c)    $(e^{-xy} - xye^{-xy}) \, dx \;\; + \;\; (2y - x^2 e^{-xy}) \, dy \;\; = \;\; 0$.

(2) Solve the following initial value problems (8 points each):

(a)    $ty' + 2y \;\; = \;\; e^t \;\; (t > 0) \qquad y(2) = 0$.

(b)    $x^2 y'' - xy' + 10y \;\; = \;\; 0 \qquad y(1) = 4, \; y'(1) = 0$.

(3) Compute the general solution of each of the following (8 points each):

(a)    $y''' - 2y'' + 10y' \;\; = \;\; 100t + 18e^t$.

(b)    $y'' + 4y' + 4y \;\; = \;\; t^{-2} e^{-2t}$.

(4) For the differential equation:

$$(3 - x)y'' \;\; + \;\; x^2 y' \;\; + \;\; (1 + x)y \;\; = \;\; 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first four nonzero terms of the solution with $y(0) = -6, y'(0) = 2$. (9 points)

(5) A tank with a capacity of 100 gallons initially contains 50 gallons of water with 10 pounds of salt in solution. Fresh water enters at a rate of 2 gallons per minute and a well-stirred mixture is pumped out at the rate of 1 gallon per minute. Compute the amount of salt (in pounds) in the tank at the first moment when the tank is filled. (8 points)

PartII Answer all sections of three (3) questions out of the five (5) questions in this part (10 points each).

(6) With $y_1$ and $y_2$ two solutions of the equation $y'' + py' + qy = 0$, assume that the Wronskian of $y_1$ and $y_2$ is not zero at $t = 0$. Show that for every pair of real numbers $A$ and $B$ there exist constants $C_1$ and $C_2$ so that $y = C_1 y_1 + C_2 y_2$ is a solution of the initial value problem consisting of the equation together with the initial conditions $y(0) = A$ and $y'(0) = B$.

(7) (a) Compute the general solution of the differential equation
$$y^{vi} + 2y''' + y \quad = \quad 0.$$

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the *method of undetermined coefficients*. Do not attempt to determine the coefficients.
$$y^{vi} + 2y''' + y \quad = \quad t^2 + e^{t/2} - e^{t/2} cos(\sqrt{3}\ t/2)$$

(8) (a) A 4 pound weight stretches a spring 6 inches downward to a position $y = 0$. Set up an initial value problem (differential equation and initial conditions) for the position $y(t)$ ($t$ seconds later) if the spring has been lifted back up the 6 inches and is released.

(b) Compute the Laplace transform $\mathcal{L}(y)(s)$ where $y$ is the solution of the initial value problem:
$$2y'' + 7y' + 6y = 0 \quad \text{with} \quad y(0) = 2 \quad \text{and} y'(0) = -2.$$

(9) For the differential equation:
$$(x^2 + 3x^3)y'' + xy' + (x^2 - 5)y \quad = \quad 0$$
Show that $x_0 = 0$ is a *regular singular point*. Compute the recursion formula for the coefficients of the series solution centered at $x_0 = 0$ which is associated with the larger root of the indicial equation and compute $a_1$ and $a_2$ when $a_0 = 1$.

(10) (a) Compute the sine series for the function $f(x) = x(1 - x)$ on the interval $[0, 1]$.

(b) Use your answer to part (a) to obtain a series solution $u(t, x)$ for the partial differential equation with $x$ in the interval $[0, 1]$ and $t > 0$:
$$u_t \quad = \quad 9u_{xx} \quad \text{with}$$
$$u(t, 0) \quad = \quad u(t, 1) \quad = \quad 0 \quad \text{for } t > 0 \quad \text{(boundary conditions)}$$
$$u(0, x) \quad = \quad x(1 - x) \quad \text{for } 0 < x < 2 \quad \text{(initial conditions)}$$