Math 391 Test 3A

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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Don't panic! I repeat, do NOT panic!
3. Complete all problems. In this exam, each non-bonus problem is worth 25 points. The weight of the bonus problems are indicated.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Do NOT commit any of the blasphemies or mistakes I mentioned in the syllabus. I will actually mete out punishment in the way I said I would. I wasn't kidding.
11. Other than that, have fun and good luck!

Remember: Find your math zen. There is nothing but form and algorithm.

1. Solve the following ODEs:

(a) $y^{(4)} + 16y = 0$

$$r^4 + 16 = 0$$

$$\Rightarrow r^4 = -16 = 16e^{i(\pi+2n\pi)}$$

$$\Rightarrow r = 2e^{i(\pi+2n\pi)/4}, \quad n = 0,1,2,3.$$

$$\Rightarrow r_1 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2}$$

$$\Rightarrow r_2 = 2e^{i3\pi/4} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\sqrt{2} + i\sqrt{2}$$

$$\Rightarrow r_3 = \sqrt{2} - i\sqrt{2}$$

$$\Rightarrow r_4 = -\sqrt{2} - i\sqrt{2}$$

$$\Rightarrow \boxed{y = c_1 e^{\sqrt{2}t}\cos\sqrt{2}t + c_2 e^{\sqrt{2}t}\sin\sqrt{2}t + c_3 e^{-\sqrt{2}t}\cos\sqrt{2}t + c_4 e^{-\sqrt{2}t}\sin\sqrt{2}t}$$

(b) (i) $y^{(6)} - 16y'' = 0$

$$r^6 - 16r^2 = 0$$

$$\Rightarrow r^2(r^4 - 16) = 0$$

$$\Rightarrow r^2(r^2-4)(r^2+4) = 0$$

$$\Rightarrow r^2(r-2)(r+2)(r+2i)(r-2i) = 0$$

$$\Rightarrow r_{1,2} = 0,\ r_3 = 2,\ r_4 = -2,\ r_5 = -2i,\ r_6 = 2i$$

$$\Rightarrow \boxed{y = c_1 + c_2 t + c_3 e^{2t} + c_4 e^{-2t} + c_5\cos 2t + c_6 \sin 2t}$$

(b) (ii) Given $y^{(6)} - 16y'' = e + 2e^{2t} - \pi\cos(2t)$. Determine the suitable form for the function $Y(t)$ with the fewest terms to obtain a particular solution, via the *method of undetermined coefficients*. You need not solve for the coefficients.

$$\boxed{Y(t) = At^2 + Bte^{2t} + Ct\cos 2t + Dt\sin 2t}$$

2. Solve the following ODEs, assume $x > 0$:
(a) $x^2y'' - 3xy' + 4y = 0$

$r(r-1) - 3r + 4 = 0$

$\Rightarrow r^2 - 4r + 4 = 0$

$\Rightarrow (r-2)^2 = 0$

$\Rightarrow r_{1,2} = 2$

$\Rightarrow \boxed{y = c_1 x^2 + c_2 x^2 \ln x}$

(b) $x^2y'' + 4xy' + 2y = 0$

$r(r-1) + 4r + 2 = 0$

$\Rightarrow r^2 + 3r + 2 = 0$

$\Rightarrow (r+2)(r+1) = 0$

$\Rightarrow r = -2,\ r = -1$

$\Rightarrow \boxed{y = c_1 x^{-2} + c_2 x^{-1}}$

(c) $x^2y'' + 5xy' + 8y = 0$

$r(r-1) + 5r + 8 = 0$

$\Rightarrow r^2 + 4r + 8 = 0$

$\Rightarrow r = \dfrac{-4 \pm \sqrt{16 - 4(8)}}{2} = \dfrac{-4 \pm 4i}{2} = -2 \pm 2i$

$\Rightarrow \boxed{y = c_1 x^{-2} \cos(2\ln x) + c_2 x^{-2} \sin(2\ln x)}$

3. Find and classify (as regular or irregular) all the singular points of

$$x^2(1-x)^2y'' + 2xy' + 4y = 0$$

For singular points: $x^2(1-x)^2 = 0$

$\Rightarrow x=0,\ x=1.$

For $x=0$

$$\lim_{x\to 0} \cancel{x} \cdot \frac{2\cancel{x}}{x^{\cancel{2}}(1-x)^2} = 2$$

$$\lim_{x\to 0} \cancel{x^2} \cdot \frac{4}{\cancel{x^2}(1-x)^2} = 4$$

$\Rightarrow$ $\boxed{x=0 \text{ is a regular singular point}}$

For $x=1$

$$\lim_{x\to 1} \cancel{(x-1)} \cdot \frac{2\cancel{x}\ (-1)}{x^{\cancel{2}}(1-x)^{\cancel{2}}} = DNE$$

$\Rightarrow$ $\boxed{x=1 \text{ is an irregular singular point}}$

4. For the differential equation:

$$(4 - x^2)y'' + (1 + x)y = 0$$

$$4y'' - x^2y'' + y + xy = 0$$

Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$, and use it to compute the first four nonzero terms of the solution with $y(0) = 2$ and $y'(0) = -1$. You may assume $a_n = 0$ for $n < 0$ to avoid some computation.

$$\Rightarrow \sum_{2}^{\infty} 4n(n-1)a_n x^{n-2} - \sum_{2}^{\infty} n(n-1)a_n x^n + \sum_{0}^{\infty} a_n x^n + \sum_{0}^{\infty} a_n x^{n+1} = 0$$

$$\Rightarrow \sum_{0}^{\infty} 4(n+2)(n+1)a_{n+2}x^n - \sum_{0}^{\infty} n(n-1)a_n x^n + \sum_{0}^{\infty} a_n x^n + \sum_{0}^{\infty} a_{n-1} x^n = 0$$

0 by last sentence...

$$\Rightarrow 4(n+2)(n+1)a_{n+2} - n(n-1)a_n + a_n + a_{n-1} = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{[n(n-1)-1]a_n - a_{n-1}}{4(n+1)(n+2)}} \rightarrow \text{Recursion formula.}$$

$$a_0 = 2$$

$$a_1 = -1$$

$$a_2 = \frac{-a_0 - \cancelto{0}{a_{-1}}}{8} = \frac{-2}{8} = -\frac{1}{4}$$

$$a_3 = \frac{-a_1 - a_0}{24} = \frac{1-2}{24} = \frac{-1}{24}$$

$$\Rightarrow \boxed{y = 2 - x - \tfrac{1}{4}x^2 - \tfrac{1}{24}x^3} \text{ up to four nonzero terms.}$$

Bonus Problems:

1. (5 points) Use the definition of the Laplace transform to derive the Laplace transform of $f(t) = t^2$.

$$L(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\infty t^2 e^{-st}\,dt$$

+	$2t$	$-1/s\, e^{-st}$
−	2	$1/s^2\, e^{-st}$
+	0	$-1/s^3\, e^{-st}$

$$= -\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\Big|_0^\infty$$

$$= \boxed{\frac{2}{s^3}}$$

2. (5 points) Use Laplace transforms to find the particular solution of: $y'' - 2y' + 2y = 0$ with $y(0) = 0$ and $y'(0) = 1$. You may consult the table of Laplace transforms attached at the end of this test. Let $\mathcal{L}y = Y$

$$\Rightarrow \mathcal{L}y' = -y(0) + s\mathcal{L}y = sY$$

$$\Rightarrow \mathcal{L}y'' = -y'(0) + s\mathcal{L}y' = -1 + s^2Y$$

$$\Rightarrow -1 + s^2Y - 2sY + 2Y = 0$$

$$\Rightarrow Y = \frac{1}{s^2 - 2s + 2}$$

$$= \frac{1}{(s-1)^2 + 1}$$

$$\Rightarrow \boxed{y = e^t \sin t}$$

3. Assume $f(x)$ is a $2L$-periodic function that has a Fourier series expansion.

(i) (5 points) What is the general form for the Fourier series expansion of $f(x)$?

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$

(ii) (4 points) Define, using integrals, the coefficients in the formula above.

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

(iii) (4 points) Suppose $f(x)$ is an odd function, then:

$a_n = 0$

$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

(iv) (4 points) Suppose $f(x)$ is an even function, then:

$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$b_n = 0$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29