Math 346 Test 3A
May 18, 2016

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Note that both sides of each page may have printed material.

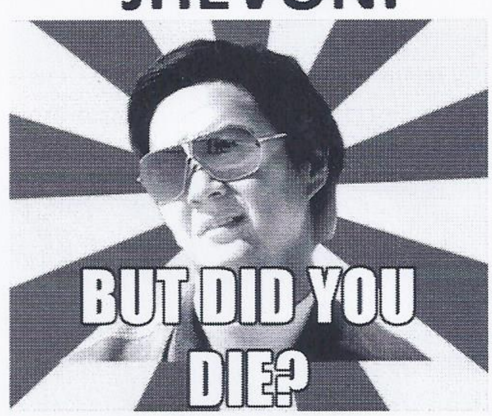
Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

STUDENT:

Jhevon chill! You make this class too hard. My GPA!!!

JHEVON:



1. Consider the matrix $A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$.

(a) (10 points) Find the eigenvalues and corresponding eigenvectors of A.

$$\begin{vmatrix} \lambda-3 & -4 \\ 1 & \lambda+2 \end{vmatrix} = 0$$

$\Rightarrow (\lambda-3)(\lambda+2)+4=0$

$\Rightarrow \lambda^2-\lambda-2=0$

$\Rightarrow (\lambda-2)(\lambda+1)=0$

$\Rightarrow \boxed{\lambda_1=2,\ \lambda_2=-1} \to$ Eigenvalues.

$\underline{\lambda_1=2}$

$$\left(\begin{array}{cc|c} 2-3 & -4 & 0 \\ 1 & 2+2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_2 \\ R_1+R_2 \end{array}$$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4t \\ t \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} t$

$\Rightarrow$ $\boxed{\begin{pmatrix} -4 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_1=2.}$

$\underline{\lambda_2=-1}$

$$\left(\begin{array}{cc|c} -1-3 & -4 & 0 \\ 1 & -1+2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_2 \\ \frac{R_1}{4}+R_2 \end{array}$$

$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$

$\Rightarrow$ $\boxed{\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ is the eigenvector corresponding to } \lambda_2=-1}$

(b) (10 points) Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Set $D=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $P=\left(\vec{\lambda}_1 \mid \vec{\lambda}_2\right)$

$\Rightarrow \boxed{D=\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},\ P=\begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}}$

(Note: $P^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$, so we have $A=PDP^{-1}$
$= \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$)

(c) (10 points) Compute A^5.

By above $A = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix}$

$\Rightarrow A^5 = \begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^5\begin{pmatrix} -1/3 & -1/3 \\ 1/3 & 4/3 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -4 \end{pmatrix}$

$= -\frac{1}{3}\begin{pmatrix} -4 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 32 & 32 \\ 1 & 4 \end{pmatrix} = \boxed{-\frac{1}{3}\begin{pmatrix} -129 & -132 \\ 33 & 36 \end{pmatrix}} = \boxed{\begin{pmatrix} 43 & 44 \\ -11 & -12 \end{pmatrix}}$

2. (10 points) Using problem 1, solve the following system for the functions $y_1(t)$ and $y_2(t)$, subject to the initial conditions $y_1(0) = 1$ and $y_2(0) = 2$.

$$\begin{Bmatrix} y_1'(t) & = & 3y_1(t) + 4y_2(t) \\ y_2'(t) & = & -y_1(t) - 2y_2(t) \end{Bmatrix}$$

We have $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

From 1 we have $\lambda_1 = 2, \vec{\lambda}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\lambda_2 = -1, \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

$$\Rightarrow \begin{aligned} y_1 &= -4c_1 e^{2t} - c_2 e^{-t} \\ y_2 &= c_1 e^{2t} + c_2 e^{-t} \end{aligned}$$

$y_1(0) = 1 \Rightarrow 1 = -4c_1 - c_2$

$y_2(0) = 2 \Rightarrow 2 = c_1 + c_2$

$3 = -3c_1$

$c_1 = -1$

$\Rightarrow c_2 = 3$

$$\Rightarrow \boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4e^{2t} - 3e^{-t} \\ -e^{2t} + 3e^{-t} \end{pmatrix}}$$

3. Let $D: P_2 \to P_2$ be the differentiation operator $D(\mathbf{p}) = p'(x)$.

(a) (15 points) Find $[D]_B$, where $B = \{2, 2-3x, 2-3x+8x^2\} = \left\{\begin{pmatrix}2\\0\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\8\end{pmatrix}\right\}$

$$[D]_B = \left[\left[D\begin{pmatrix}2\\0\\0\end{pmatrix}\right]_B \middle| \left[D\begin{pmatrix}2\\-3\\0\end{pmatrix}\right]_B \middle| \left[D\begin{pmatrix}2\\-3\\8\end{pmatrix}\right]_B\right]$$

$$= \left[\begin{pmatrix}0\\0\\0\end{pmatrix}_B \middle| \begin{pmatrix}-3\\0\\0\end{pmatrix}_B \middle| \begin{pmatrix}-3\\16\\0\end{pmatrix}_B\right]$$

$$= \boxed{\begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}}$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}_B$: $\begin{pmatrix}0\\0\\0\end{pmatrix}_B = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\begin{pmatrix}-3\\0\\0\end{pmatrix}_B$: $\begin{pmatrix}-3\\0\\0\end{pmatrix} = c_1\begin{pmatrix}2\\0\\0\end{pmatrix} + c_2\begin{pmatrix}2\\-3\\0\end{pmatrix} + c_3\begin{pmatrix}2\\-3\\8\end{pmatrix}$

$\Rightarrow \begin{pmatrix}-3\\0\\0\end{pmatrix}_B = \begin{pmatrix}-3/2\\0\\0\end{pmatrix}$

$\begin{pmatrix}-3\\16\\0\end{pmatrix}_B$: $\begin{pmatrix}-3\\16\\0\end{pmatrix} = c_1\begin{pmatrix}2\\0\\0\end{pmatrix} + c_2\begin{pmatrix}2\\-3\\0\end{pmatrix} + c_3\begin{pmatrix}2\\-3\\8\end{pmatrix}$

$$\rightarrow \left(\begin{array}{ccc|c} 2 & 2 & 2 & -3\\ 0 & -3 & -3 & 16\\ 0 & 0 & 8 & 0\end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -3/2\\ 0 & 1 & 1 & -16/3\\ 0 & 0 & 1 & 0\end{array}\right) \begin{array}{l} R_1/2\\ R_2/-3\\ R_3/8\end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 23/6\\ 0 & 1 & 0 & -16/3\\ 0 & 0 & 1 & 0\end{array}\right) \begin{array}{l} R_1 - R_2\\ R_2 - R_3\end{array}$$

$\Rightarrow \begin{pmatrix}-3\\16\\0\end{pmatrix}_B = \begin{pmatrix}23/6\\-16/3\\0\end{pmatrix}$

(b) (10 points) Use part (a) to compute $D(6 - 6x + 24x^2)$.

$$\begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}6\\-6\\24\end{pmatrix}_B = \begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 6\\ 0 & -3 & -3 & -6\\ 0 & 0 & 8 & 24\end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3\\ 0 & 1 & 1 & 2\\ 0 & 0 & 1 & 3\end{array}\right)$$

$$= \boxed{\begin{pmatrix}13\\-16\\0\end{pmatrix}} = 13\begin{pmatrix}2\\0\\0\end{pmatrix} + (-16)\begin{pmatrix}2\\-3\\0\end{pmatrix} = \boxed{\begin{pmatrix}-6\\48\\0\end{pmatrix}}$$

In base B.

In standard basés.

(c) (15 points) Compute $[D(6 - 6x + 24x^2)]_B$

$$[D(6-6x+24x^2)]_B = [D]_B[6-6x+24x^2]_B$$

$$= \begin{pmatrix}0 & -3/2 & 23/6\\ 0 & 0 & -16/3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}13\\-16\\0\end{pmatrix}}$$