**Math 346 Test 2B**

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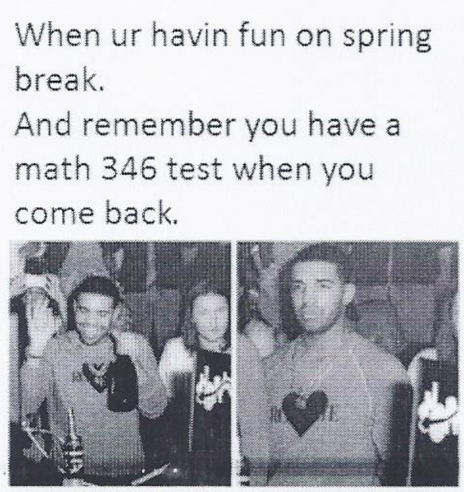
**Note that both sides of each page may have printed material.**

**Instructions:**

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! $x^2$ and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!



1. Consider bases $B = \{\vec{u}, \vec{v}\}$ and $B' = \{\vec{u}', \vec{v}'\}$ for $\mathbb{R}^2$ where $\vec{u} = \binom{2}{2}$, $\vec{v} = \binom{4}{-1}$, $\vec{u}' = \binom{1}{3}$, $\vec{v}' = \binom{-1}{-1}$.

(a) (5 points) Find the transition matrix $P_{B' \to B}$.

Using $\left(\begin{array}{c|c} \text{new basis} & \text{old basis} \end{array}\right)$

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 2 & -1 & 3 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & -1 \\ 0 & 5 & -2 & 0 \end{array}\right) \begin{array}{l} R_1 \\ R_1 - R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 10 & 0 & 13 & -5 \\ 0 & 1 & -2/5 & 0 \end{array}\right) \begin{array}{l} 5R_1 - 4R_2 \\ R_2/5 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 13/10 & -1/2 \\ 0 & 1 & -2/5 & 0 \end{array}\right) \begin{array}{l} R_1/10 \\ \phantom{R} \end{array}$$

$$\Rightarrow \boxed{P_{B' \to B} = \begin{pmatrix} 13/10 & -1/2 \\ -2/5 & 0 \end{pmatrix}}$$

(b) (5 points) Find the transition matrix $P_{B \to B'}$.

$$\left(\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 3 & -1 & 2 & -1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 2 & 4 \\ 0 & -2 & 4 & 13 \end{array}\right) \begin{array}{l} R_1 \\ 3R_1 - R_2 \end{array}$$

$$\left(\begin{array}{cc|cc} 2 & 0 & 0 & -5 \\ 0 & 1 & -2 & -13/2 \end{array}\right) \begin{array}{l} 2R_1 - R_2 \\ R_2/-2 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & -5/2 \\ 0 & 1 & -2 & -13/2 \end{array}\right) \begin{array}{l} R_1/2 \\ \phantom{R} \end{array}$$

$$\Rightarrow \boxed{P_{B \to B'} = \begin{pmatrix} 0 & -5/2 \\ -2 & -13/2 \end{pmatrix}}$$

↳ Note, you could also use $P_{B \to B'} = \left(P_{B' \to B}\right)^{-1}$

(c) (5 points) Compute the coordinate vector $[\vec{w}]_B$ where $\vec{w} = \binom{3}{-5}$.

Set $\binom{3}{-5} = c_1\binom{2}{2} + c_2\binom{4}{-1} = \binom{2c_1 + 4c_2}{2c_1 - c_2}$

$$\Rightarrow \left(\begin{array}{cc|c} 2 & 4 & 3 \\ 2 & -1 & -5 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & 4 & 3 \\ 0 & 5 & 8 \end{array}\right) \begin{array}{l} R_1 \\ R_1 - R_2 \end{array}$$

$$\left(\begin{array}{cc|c} 10 & 0 & -17 \\ 0 & 1 & 8/5 \end{array}\right) \begin{array}{l} 5R_1 - 4R_2 \\ R_2/5 \end{array}$$

$$\left(\begin{array}{cc|c} 1 & 0 & -17/10 \\ 0 & 1 & 8/5 \end{array}\right) \begin{array}{l} R_1/10 \\ R_2 \end{array}$$

$$\Rightarrow \boxed{[\vec{w}]_B = \begin{pmatrix} -17/10 \\ 8/5 \end{pmatrix}}$$

(d) (5 points) Use either part (a) or part (b) to compute $[\vec{w}]_{B'}$

$$[\vec{w}]_{B'} = P_{B \to B'}[\vec{w}]_B = \begin{pmatrix} 0 & -5/2 \\ -2 & -13/2 \end{pmatrix}\begin{pmatrix} -17/10 \\ 8/5 \end{pmatrix} = \boxed{\begin{pmatrix} -4 \\ -7 \end{pmatrix}}$$

2. (a) (8 points) Prove that $W = \{A \in M_{22} : A^T = A\}$ is a subspace of $M_{22}$.

Pf: The vectors in W are of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

① Closure under addition: Let $\vec{x} = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix}, \vec{y} = \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} \in W$.

$\Rightarrow \vec{x}+\vec{y} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & c_1+c_2 \end{pmatrix}$. Since $(\vec{x}+\vec{y})^T = \vec{x}+\vec{y}$, $\vec{x}+\vec{y} \in W$.

② Closure under scalar multiplication: Let $\vec{x} = \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \in W$.

$\Rightarrow k\vec{x} = \begin{pmatrix} ka_1 & kb_1 \\ kb_1 & kc_1 \end{pmatrix}$. Since $(k\vec{x})^T = k\vec{x}$, $k\vec{x} \in W$. ■

(b) (12 points) Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that $B = \{A_1, A_2, A_3\}$ is a basis for $W$ defined above.

Pf: Set $c_1 A_1 + c_2 A_2 + c_3 A_3 = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\Rightarrow c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\Rightarrow \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

$\Rightarrow c_1 = a,\ c_2 = c,\ c_3 = b$ — ①

$\Rightarrow$ $\boxed{\{A_1, A_2, A_3\} \text{ spans } W.}$

Moreover, if we choose $a, b, c = 0$, then we have that $c_1 A_1 + c_2 A_2 + c_3 A_3 = \vec{0}$ has only the trivial sol'n, by equation ①.

$\Rightarrow$ $\boxed{A_1, A_2, A_3 \text{ are linearly independent.}}$ ■

3. Let $T: V \to W$ be a linear transformation.

(a) (6 points) Define $\ker T$ and prove that it is a subspace of $V$.

DEF: $\ker T = \{\vec{x} \in V \mid T(\vec{x}) = \vec{0}\}$.

Pf: $\ker T$ is a subspace of $V$

① Closure under addition: If $\vec{x}, \vec{y} \in \ker T$, then $T(\vec{x}+\vec{y}) \overset{\text{Since } T \text{ is a lin. trans.}}{=} T(\vec{x}) + T(\vec{y})$

$$= \vec{0} + \vec{0}$$
$$= \vec{0}$$
$$\Rightarrow \vec{x}+\vec{y} \in \ker T.$$

② Closure under scalar multiplication:

If $\vec{x} \in \ker T$, then $T(k\vec{x}) = kT(\vec{x})$

$$= k\vec{0}$$
$$= \vec{0}$$
$$\Rightarrow k\vec{x} \in \ker T. \quad \blacksquare$$

(b) (6 points) Define $R(T)$ and prove that it is a subspace of $W$.

DEF: $R(T) = \{\vec{y} \in W \mid T(\vec{x}) = \vec{y}$ for some $\vec{x} \in V\}$.

Pf: $R(T)$ is a subspace of $W$

① Closure under addition: If $\vec{y}_1, \vec{y}_2 \in R(T)$, then there are $\vec{x}_1, \vec{x}_2 \in V$ such that $T(\vec{x}_1) = \vec{y}_1$ and $T(\vec{x}_2) = \vec{y}_2$. Since $V$ is a vector space, $\vec{x}_1 + \vec{x}_2 \in V$ and $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2) = \vec{y}_1 + \vec{y}_2 \Rightarrow \vec{y}_1 + \vec{y}_2 \in R(T)$.

② Closure under scalar multiplication: If $\vec{y} \in R(T)$, then there exists $\vec{x} \in V$ such that $T(\vec{x}) = \vec{y}$. Since $V$ is a vector space, $k\vec{x} \in V$ and $T(k\vec{x}) = kT(\vec{x}) = k\vec{y} \Rightarrow k\vec{y} \in R(T)$. $\blacksquare$

(c) (8 points) Suppose $T$ is represented by a square matrix $A$. Prove that the following are equivalent:

(i) $\ker T = \{\vec{0}\}$

(ii) $T$ is one-to-one

(iii) $A$ is invertible

Pf: (i) $\Rightarrow$ (ii): Assume $\ker T = \{\vec{0}\}$. Suppose $T(\vec{x}) = T(\vec{y})$. Then $T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = \vec{0}$. Since $\ker T = \{\vec{0}\}$, we must have $\vec{x} - \vec{y} = \vec{0} \Rightarrow \vec{x} = \vec{y} \Rightarrow T$ is 1-1.

(ii) $\Rightarrow$ (iii): Assume $T$ is 1-1, and $T(\vec{x}) = A\vec{x}$. If $T(\vec{x}) = \vec{y}$, then $A\vec{x} = \vec{y}$ and there is only one $\vec{x} \in \text{dom}(T)$ that maps to $\vec{y}$. $\Rightarrow A\vec{x} = \vec{y}$ has a unique soln. and so $A$ is invertible.

(iii) $\Rightarrow$ (i): Assume $A$ is invertible. Then $A\vec{x} = \vec{0}$ has only the trivial soln. That is, if $T(\vec{x}) = \vec{0}$, then $\vec{x}$ must be $\vec{0} \Rightarrow \ker T = \{\vec{0}\}$. $\blacksquare$