

Name:

JHEVON SMITH

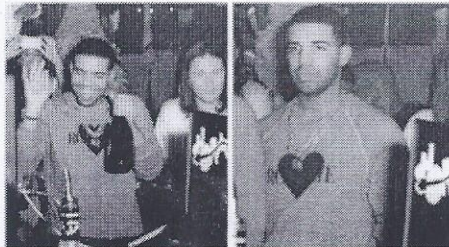
Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Scientific calculators are needed, but you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

When ur havin fun on spring
break.

And remember you have a
math 346 test when you
come back.



1. Let $T: V \rightarrow W$ be a linear transformation.

(a) (6 points) Define $\ker T$ and prove that it is a subspace of V .

DEF: $\ker T = \{ \vec{x} \in V \mid T(\vec{x}) = \vec{0} \}$.

Pf: $\ker T$ is a subspace of V .

① Closure under addition: If $\vec{x}, \vec{y} \in \ker T$, then $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
 $= \vec{0} + \vec{0}$
 $= \vec{0}$
 $\Rightarrow \vec{x} + \vec{y} \in \ker T$. since T is a lin. trans.

② Closure under scalar multiplication:

If $\vec{x} \in \ker T$, then $T(k\vec{x}) = kT(\vec{x})$
 $= k\vec{0}$
 $= \vec{0}$
 $\Rightarrow k\vec{x} \in \ker T$. ■

(b) (6 points) Define $R(T)$ and prove that it is a subspace of W .

DEF: $R(T) = \{ \vec{y} \in W \mid T(\vec{x}) = \vec{y} \text{ for some } \vec{x} \in V \}$.

Pf: $R(T)$ is a subspace of W .

① Closure under addition: If $\vec{y}_1, \vec{y}_2 \in R(T)$, then there are $\vec{x}_1, \vec{x}_2 \in V$ such that $T(\vec{x}_1) = \vec{y}_1$ and $T(\vec{x}_2) = \vec{y}_2$. So, since V is a vector space, $\vec{x}_1 + \vec{x}_2 \in V$ and $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2) = \vec{y}_1 + \vec{y}_2 \Rightarrow \vec{y}_1 + \vec{y}_2 \in R(T)$.

② Closure under scalar multiplication: If $\vec{y} \in R(T)$, then there exists $\vec{x} \in V$ such that $T(\vec{x}) = \vec{y}$. Since V is a vector space, $k\vec{x} \in V$ and $T(k\vec{x}) = kT(\vec{x}) = k\vec{y} \Rightarrow k\vec{y} \in R(T)$. ■

(c) (8 points) Suppose T is represented by a square matrix A . Prove that the following are equivalent:

- (i) $\ker T = \{ \vec{0} \}$
- (ii) T is one-to-one
- (iii) A is invertible

Pf: (i) \Rightarrow (ii): Assume $\ker T = \{ \vec{0} \}$. Suppose $T(\vec{x}) = T(\vec{y})$. Then $T(\vec{x}) - T(\vec{y}) = T(\vec{x} - \vec{y}) = \vec{0}$. Since $\ker T = \{ \vec{0} \}$, we have $\vec{x} - \vec{y} = \vec{0} \Rightarrow \vec{x} = \vec{y}$.
 $\Rightarrow T$ is 1-1.

(ii) \Rightarrow (iii): Assume T is 1-1, and $T(\vec{x}) = A\vec{x}$. If $T(\vec{x}) = \vec{y}$, then $A\vec{x} = \vec{y}$ and there is only one $\vec{x} \in \text{dom}(T)$ that maps to \vec{y} . $\Rightarrow A\vec{x} = \vec{y}$ has a unique soln and so A is invertible.

(iii) \Rightarrow (i): Assume A is invertible. Then $A\vec{x} = \vec{0}$ has only the trivial soln. That is, if $T(\vec{x}) = \vec{0}$, then \vec{x} must be $\vec{0} \Rightarrow \ker(T) = \{ \vec{0} \}$. ■

2. Let $A = \begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 2 & 4 & 7 & 7 & 0 \\ 3 & 6 & 9 & 9 & -1 \\ 1 & 2 & 4 & 4 & 1 \end{pmatrix}$

(a) (8 points) Find a basis for the column space of A

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 2 & 4 & 7 & 7 & 0 \\ 3 & 6 & 9 & 9 & -1 \\ 1 & 2 & 4 & 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \cdot -1 \\ R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} R_3 - R_4$$

$$\Rightarrow \text{Basis for } \text{col}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \\ 9 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(b) (6 points) Find a basis for the null space of A

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & | & 0 \\ 0 & 0 & 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$x_2 = t, \quad x_4 = r$$

$$R_3: x_5 = 0$$

$$R_2: x_3 = -x_4 = -r$$

$$R_1: x_1 = -2x_2 - 3x_3 - 3x_4 = -2t + 0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ -r \\ r \\ 0 \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{Basis for } \text{Null}(A) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(c) (4 points) Find $\text{rank}(A)$ and $\text{Nullity}(A)$.

$$\text{Rank}(A) = \# \text{ of pivots} = 3$$

$$\text{Nullity}(A) = 2 \rightarrow \# \text{ of columns} - \text{rank}, \text{ or } \dim(\text{Null}(A)), \text{ or } \# \text{ of free variables in (b).}$$

(d) (2 points) The correct answer to (c) is an example of a general principle. What is this principle?

The Dimension Theorem. (for Matrices).

