SAMPLE 212 FINAL EXAM SOLUTIONS

1. (a) Evaluate: $\int 5^x \tan(5^x)\, dx$

(b) Find $\frac{d}{dx}\left[\sqrt{\log_3 x}\right]$

(c) Does the graph of $x + x^3 + y^2 + y^4 + z^4 + z^5 = 1$ have symmetry about

(i) the x-axis? (ii) the xz-plane? (iii) the origin?

(d) What is the value of $\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \sin(x^3)\, dx$. Explain why.

————SOLUTION————

(a) Substitute $u = 5^x$; $du = (\ln 5)5^x\, dx$:

$$\int 5^x \tan(5^x)\, dx = \int \tan u(\frac{1}{\ln 5}\, du) = \frac{\ln|\sec(5^x)|}{\ln 5} + C$$

(b)

$$\frac{d}{dx}\left[\sqrt{\log_3 x}\right] = \frac{1}{2\sqrt{\log_3 x}} \cdot \left(\frac{\ln x}{\ln 3}\right)'$$

$$= \frac{1}{2\sqrt{\log_3 x}} \frac{1}{\ln 3} \frac{1}{x} = \frac{1}{2x(\ln 3)\sqrt{\log_3 x}}$$

OR

$$= \frac{1}{2\sqrt{(\ln x)/\ln 3}} \frac{1}{\ln 3} \frac{1}{x} = \frac{1}{2x(\sqrt{\ln 3})\sqrt{\ln x}}$$

(c) (i) Symmetry about the x-axis: replace y and z by their negatives; this is not an equivalent equation, so the answer is No.

(ii) xz-plane: Replace y by $-y$. Yes.

(iii) origin: Replace all variables by their negatives: No

(d) This is an integral of an odd function over symmetric limits. The value is 0.

2. (a) Let the function $y(x)$ be the solution to the differential equation $y' = \dfrac{12x^3 + 12x^2}{y^2 e^{y^3}}$ for which $y(1) = 0$. Find $y(x)$ explicitly as a function of x.

(b) Find $\int 4x[\sec(x^2 + 2)]\, dx$.

SOLUTION

(a) Substitute $u = y^3$; $du = 3y^2\, dy$, let $C = 3C'$, and substitute $x = 1$:

$$\frac{dy}{dx} = \frac{12x^3 + 12x^2}{y^2 e^{y^3}}$$
$$\int y^2 e^{y^3}\, dy = \int 12x^3 + 12x^2\, dx$$
$$\int e^u \left(\frac{1}{3} du\right) = \frac{1}{3} e^{y^3} = 3x^4 + 4x^3 + C'$$
$$y^3 = \ln(9x^4 + 12x^3 + C)$$
$$y = \sqrt[3]{\ln(9x^4 + 12x^3 + C)}$$
$$0 = y(1) = \sqrt[3]{\ln(21 + C)}$$
$$\ln(21 + C) = 0;\ 21 + C = 1$$
$$y = \sqrt[3]{\ln(9x^4 + 12x^3 - 20)}$$

(b) Let $u = x^2 + 2$;

$$\int 4x[\sec(x^2 + 2)]\, dx = \int \sec u (2\, du)$$
$$= 2 \ln|\sec(x^2 + 2) + \tan(x^2 + 2)| + C$$

3. (a) A package initially with a temperature of 150° F is placed in a room maintained at 70° F. Assume the temperature difference between package and the room t hours after the package was initially placed, $(\Delta T)(t) = T(t) - 70$, where $T(t)$ is the temperature of the package t hours after being placed in the room, satisfies an exponential decay law. The temperature of the package 2 hours after being placed in the room is 86° F.

(i) Find the function $(\Delta T)(t)$.

(ii) Find the temperature, expressed as a rational number (i.e., a quotient of integers), of the package after 4 hours.

(iii) How long does it take for the package to reach 71°?

(b) Evaluate: $\int \sec(3x+4)\,dx$

———SOLUTION———

$$\text{(a,i)}\ (\Delta T)(t) = (\Delta T)_0 \left(\frac{(\Delta T)(2)}{(\Delta T)(0)}\right)^{t/(2-0)}$$

$$= 80\left(\frac{16}{80}\right)^{t/2} = 80\left(\frac{1}{5}\right)^{t/2} = 80e^{-\frac{\ln 5}{2}t}$$

(ii) Using base $\frac{1}{5}$ (easiest): $(\Delta T)(4) = 80\left(\frac{1}{5}\right)^2 = \frac{16}{5}$.
With e: $(\Delta T)(4) = 80e^{-2\ln 5} = 80e^{\ln(1/25)} = \frac{80}{25} = \frac{16}{5}$.
$T(4) = 70 + \frac{16}{5} = 73.2$

$$\text{(iii)}\ 1 = (\Delta T)(t) = 80\left(\frac{1}{5}\right)^{t/2}$$

$$0 = \ln 80 + \frac{t}{2}\ln(1/5)$$

$$t = \frac{-2\ln 80}{\ln 1/5} = \frac{2\ln 80}{\ln 5}$$

(b) Let $u = 3x+4$; $\int \sec(3x+4)\,dx = \frac{1}{3}\int \sec u\,du =$
$\frac{1}{3}\ln|\sec(3x+4) + \tan(3x+4)| + C$

4. (a) Evaluate: $\displaystyle\int_0^1 \arctan x\, dx$

(b) Evaluate: $\displaystyle\int_1^2 \frac{x^2-2x-4}{x^3+2x^2}\, dx$

———SOLUTION———

(a) Integrate $1\cdot\arctan x$ by parts, with 1 being the factor integrated, and substitute $u = 1+x^2$:

$$\begin{aligned}\int_0^1 1\cdot\arctan x\, dx &= x\arctan x\Big|_0^1 - \int_0^1 x\frac{1}{1+x^2}\, dx\\ &= \frac{\pi}{4} - \int_1^2 \frac{1}{u}\left(\frac{1}{2}du\right) = \frac{\pi}{4} - \frac{1}{2}\ln u\Big|_1^2\\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2\end{aligned}$$

(b)

$$\begin{aligned}&\int_1^2 \frac{x^2-2x-4}{x^3+2x^2}\, dx\\ &= \int_1^2 \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}\, dx\\ &= \int_1^2 -\frac{2}{x^2} + \frac{1}{x+2}\, dx\\ &= \left(\frac{2}{x} + \ln|x+2|\right)\Big|_1^2\\ &= -1 + \ln 4 - \ln 3\\ &= -1 + \ln\frac{4}{3}\end{aligned}$$

$$\begin{aligned}&x^2-2x-4\\ &= A(x^2+2x) + B(x+2) + Cx^2\\ &= (A+C)x^2 + (2A+B)x + 2B\end{aligned}$$

$$2B = -4 \longrightarrow B = -2$$

$$2A + B = -2 \longrightarrow A = 0$$

$$A + C = 1 \longrightarrow C = 1$$

5. (a) Evaluate: $\int (\tan x + \sin^2 x \cos x) \sin 2x \, dx$

[Suggestion: Use the double angle formula for $\sin 2x$.]

(b) Evaluate: $\int \sqrt{1 - 4x^2} \, dx$

—————SOLUTION—————

(a) Let $u = \cos x$:

$$\int (\frac{\sin x}{\cos x} + \sin^2 x \cos x) 2 \sin x \cos x \, dx$$

$$= \int 2 \sin^2 x + 2 \sin^3 x \cos^2 x \, dx$$

$$= \int 1 - \cos 2x \, dx + \int 2(1 - \cos^2 x) \cos^2 x \sin x \, dx$$

$$= x - \frac{1}{2} \sin 2x + \int 2(u^4 - u^2) \, du$$

$$= x - \frac{1}{2} \sin 2x + \frac{2 \cos^5 x}{5} - \frac{2 \cos^3 x}{3} + C$$

(b) $\int \sqrt{1 - 4x^2} \, dx$

$$= \int \cos \theta (\frac{1}{2} \cos \theta) \, d\theta$$

$$= \frac{1}{4} \int 1 + \cos 2\theta \, d\theta$$

$$= \frac{1}{4} (\theta + \frac{1}{2} \sin 2\theta)$$

$$= \frac{1}{4} \arcsin 2x + \frac{1}{4} \sin \theta \cos \theta$$

$$= \frac{1}{4} \arcsin 2x + \frac{1}{4} (2x) \sqrt{1 - 4x^2}$$

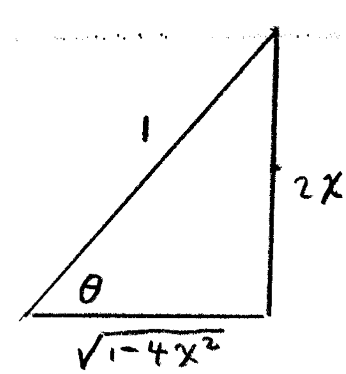
$$= \frac{1}{4} \arcsin 2x + \frac{1}{2} x \sqrt{1 - 4x^2} + C$$

$2x = \sin \theta$

$x = \frac{1}{2} \sin \theta$

$\arcsin 2x = \theta$

$dx = \frac{1}{2} \cos \theta \, d\theta$



6. (a) Which of the following improper integrals is/are convergent? Show why.

(i) $\displaystyle\int_0^1 \frac{\ln(x+2)}{x\sqrt{x}}\,dx$ (ii) $\displaystyle\int_1^\infty xe^{-2x}\,dx$

(iii) $\displaystyle\int_1^\infty \frac{x\arctan x}{\sqrt[3]{x^7+7}}\,dx$

(b) Find $\displaystyle\lim_{x\to\infty} e^{\frac{x+\ln x}{\sqrt{x^2+x+1}}}$.

———SOLUTION———

(a, i) $\displaystyle\int_0^1 \frac{\ln(x+2)}{x\sqrt{x}}\,dx \geq \int_0^1 \frac{\ln 2}{x^{3/2}}\,dx = \infty.$ Div

(ii) $\displaystyle\int_1^\infty xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x}\Big|_1^\infty - \int_1^\infty -\frac{1}{2}e^{-2x}\,dx$

$\displaystyle= -\frac{1}{2}\frac{x}{e^{2x}}\Big|_1^\infty - \frac{1}{4}e^{-2x}\Big|_1^\infty$

$\displaystyle= \frac{1}{2e^2} + \frac{1}{4e^2} = \frac{3}{4e^2}.$ Convergent

(iii) $\displaystyle\int_1^\infty \frac{x\arctan x}{\sqrt[3]{x^7+7}}\,dx \leq \int_1^\infty \frac{x\pi/2}{\sqrt[3]{x^7+7}}\,dx,$
and the second integral is convergent by the comparison with
$\displaystyle\int_1^\infty \frac{1}{x^{4/3}}\,dx.$

(b) $\displaystyle\lim_{x\to\infty} e^{\frac{x+\ln x}{\sqrt{x^2+x+1}}} = \lim_{x\to\infty} e^{\frac{x}{\sqrt{x^2+x+1}}+\frac{\ln x}{\sqrt{x^2+x+1}}} = e^{1+0} = e.$

7. State, for each series, whether it converges absolutely, converges conditionally or diverges. Name a test which supports each conclusion and show the work to apply the test.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(1+\frac{1}{n}+\frac{1}{n^2}\right)^n$ (b) $\displaystyle\sum_{n=0}^{\infty}\frac{n^2 3^n \ln(n+2)}{2^{2n+2}}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n^2+3)}{n^3+4}$

———SOLUTION———

For each series, a_n will denote the nth term of the series.

(a) Since $a_n \geq 1$ for all n, $\lim a_n \neq 0$ and the series diverges by the Test for Divergence.

(b) By L'Hopital's Rule, as n and x go to infinity, $\lim \frac{\ln(n+3)}{\ln(n+2)} = \lim \frac{\ln(x+3)}{\ln(x+2)} = \lim \frac{1/(x+3)}{1/(x+2)} \longrightarrow 1$. So

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^2}{n^2}\frac{\ln(n+3)}{\ln(n+2)}\frac{3^{n+1}}{3^n}\frac{2^{2n+2}}{2^{2n+4}} \longrightarrow 1\cdot 1\cdot 3\cdot \tfrac{1}{2^2} = \tfrac{3}{4} < 1.$$

Thus, the series converges absolutely by the Ratio Test.

(c) Clearly a_n is alternating and converges to zero. Since $\left(\dfrac{x^2+3}{x^3+4}\right)' = -\frac{x(x^3+9x-8)}{(x^3+4)^2} < 0$ for $x \geq 1$, the terms form a decreasing sequence, and the Alternating Series Test says the series is convergent. The series $\sum |a_n|$ is divergent by the Limit Comparison Test with $b_n = \dfrac{1}{n}$. Thus, $\sum a_n$ converges conditionally.

8. (a) Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n(x+1)^n}{5^n \sqrt{n^2+4}}$.

Remember to check the endpoints if applicable.

(b) Sketch the graph of the polar equation $r = 3 + 2\sin\theta$, and find the area which is both inside the graph and above the x-axis.

———SOLUTION———

(a) The center of the series is $x_0 = -1$, and the nth coefficient is $c_n = \frac{n}{5^n\sqrt{n^2+4}}$.

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{(n+1)5^n\sqrt{n^2+4}}{n5^{n+1}\sqrt{(n+1)^2+4}} \longrightarrow \frac{1}{5},$$

so the radius of convergence is $R = 5$, and the series converges for x in the open interval $(-1-5, -1+5) = (-6, 4)$.

At $x = -6$ the series is $\sum \frac{n(-1)^n}{\sqrt{n^2+4}}$ which is divergent.

At $x = 4$ the series is $\sum \frac{n}{\sqrt{n^2+4}}$ which is divergent.

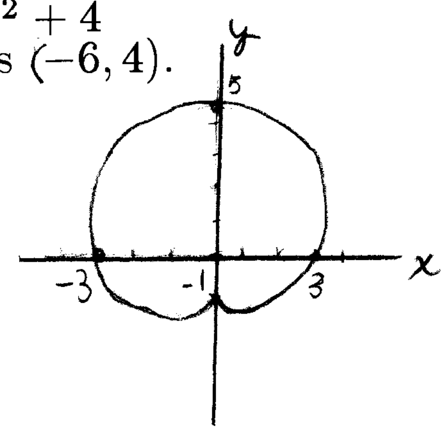
The interval of convergence is $(-6, 4)$.

(b) $r(0) = 3$

$r(\pi/2) = 5$

$r(\pi) = 3$

$r(3\pi/2) = 1$



$$\begin{aligned} A &= \frac{1}{2}\int_0^\pi (3 + 2\sin\theta)^2 \, d\theta \\ &= \frac{1}{2}\int_0^\pi 9 + 12\sin\theta + 4\sin^2\theta \, d\theta \\ &= \frac{9\pi}{2} - 6\cos\theta|_0^\pi + \int_0^\pi 1 - \cos 2\theta \, d\theta \\ &= \frac{9\pi}{2} + 12 + \pi = \frac{11\pi}{2} + 12. \end{aligned}$$

9. (a) Find the first four nonzero terms of the Maclaurin series (i.e., power series centered at 0) for the function

$$g(x) = e^{-x^2}.$$

(b) Find the sum of the first four terms of the Maclaurin series for the derivative of the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^3}$.

(c) Use the answer in part (b) to approximate $f'(-\frac{1}{2})$ with an with an error of less than .01.

———SOLUTION———

(a) Replace x by $-x^2$ in the series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} \pm \ldots$$

(b) $$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^3} = 1 + \frac{x}{2^3} + \frac{x^2}{3^3} + \frac{x^3}{4^3} + \frac{x^4}{5^3} + \ldots$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)^3} = \frac{1}{2^3} + \frac{2x}{3^3} + \frac{3x^2}{4^3} + \frac{4x^3}{5^3} + \ldots$$

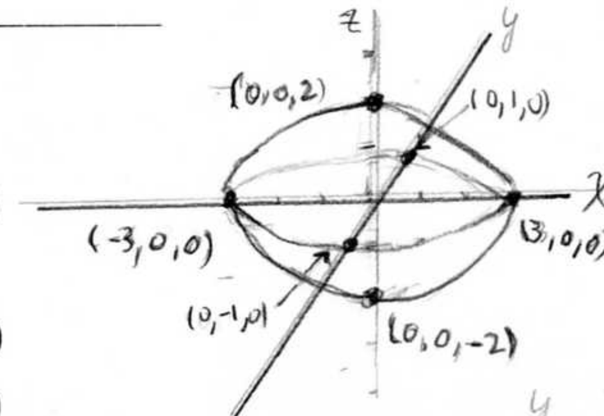
$$f'(-\frac{1}{2}) = \boxed{\frac{1}{2^3} - \frac{2}{3^3 2} + \frac{3}{4^3 2^2}} - \frac{4}{5^3 2^3} + \ldots$$

The boxed sum has error not more than $\frac{4}{5^3 2^3} = \frac{1}{250} < .01$.

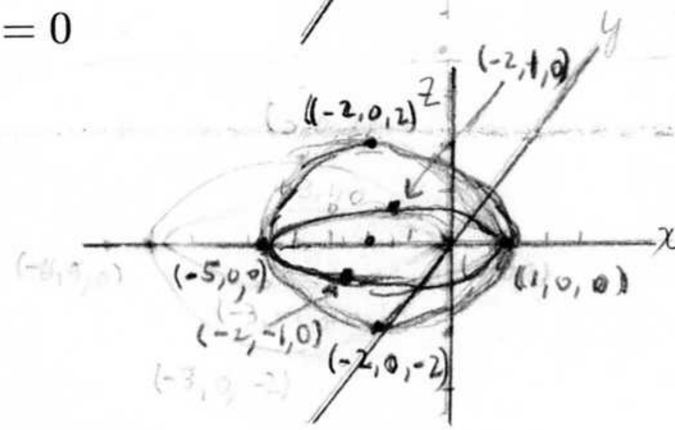
10. (a) Graph $4x^2 + 36y^2 + 9z^2 + 16x - 20 = 0$, and graph the trace of the answer to (a) in the xy-plane.

(b) Sketch the portion of the graph of $y = 1 - x^2$ which is in the first octant.

———SOLUTION———



(a)
$$4x^2 + 16x + 36y^2 + 9z^2 - 20 = 0$$
$$4[x^2 + 4x] + 36y^2 + 9z^2 - 20 = 0$$
$$4[(x+2)^2 - 4] + 36y^2 + 9z^2 - 20 = 0$$
$$4(x+2)^2 - 16 + 36y^2 + 9z^2 - 20 = 0$$
$$4(x+2)^2 + 36y^2 + 9z^2 = 36$$
$$\frac{(x+2)^2}{9} + y^2 + \frac{z^2}{4} = 0$$



(b) The graph is the cylinder obtained by moving the graph of $y = 1-x^2$, $x \geq 0$ in the z-direction.