Section 10.1

Limits

We summarize some of the material in Section 10.1 and some new material.

We will use the terminology

"a sequence of statements $A(n)$, is said to be *eventually* true if there is some n_0 such that $A(n)$ true for $n \geq n_0$."

We say $\lim_{n \to k} a_n = L$, for a real number L, if a_n is eventually in the interval $(L - \epsilon, L + \epsilon)$ for every positive number ϵ.

We say $\lim_{n \to k} a_n = \infty$ $(-\infty)$, if $a_n > (<) M$ eventually for every real number M.

For example, if $a_n = n^3 + n + 1$ and M is any real number, then $a_n > M$ eventually, so $\lim_{n \to \infty} a_n = \infty$.

Since the distance between integers n and k is less than 1 only if $n = k$, $\lim_{n \to k} a_n = L$ if and only if, for some k, $L = a_k$ and eventually $a_n = a_k$. This is not a useful result, so only limits of a sequence a_n as n goes to infinity are considered, and just lim **will be used to mean the limit as the variable goes to infinity.**

We will frequently be computing limits of sequences for the material in Chapter 10, and the following three results will be useful:

1. $\lim a_n = 0$ if and only if $\lim |a_n| = 0$, because $|a_n - 0| = |(|a_n| - 0)|$.

2. If a_n is a sequence and $f(x)$ is a function defined for all real numbers x greater than or equal to 1 and $a_n = f(n)$ for all positive integers n and the $\lim f(x)$ exists, then $\lim a_n$ exists and

$$\lim a_n = \lim f(x).$$

3. A bounded monotone sequence is convergent.
Example 1: $\lim \frac{n}{e^n} = \lim \frac{x}{e^x} = 0$.

We sometimes call a sequence a discrete function, more precisely discretely varying, function, and call a function that varies over all real numbers in some interval a continuously varying function.

Because of the relationship between discretely and continuously varying functions, the material in the supplementary notes for Section 8.8 is repeated here for convenient reference:

Rate of Growth of Functions

The material presented here is similar to that in Section 7.4. It is useful when applying the Limit Comparison for improper integrals, and a discrete analogue of this material will be essential for much of what is done in Chapter 10 on infinite series.

By the phrase "$f = o(g)$ at a" we mean $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. Usually, we will be interested in $a = \infty$ and functions f and g for which $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Our intuitive interpretation of $f = o(g)$ is that the function g is getting large faster than f as $x \to \infty$.

From now on lim will mean $\lim_{x\to\infty}$.

Example 1: If $0 < m < n$, then $x^m = o(x^n)$, i.e.,

$$\lim \frac{x^m}{x^n} = \lim \frac{1}{x^{n-m}} = 0.$$

For a more specific example, x^2 and x^3 both get large as x becomes large; 100^2 is ten thousand, quite large, but 100^3 is a million, much larger.

The notation used here, instead of $f = o(g)$, is $f << g$. This is done to call attention to the fact that the relation is transitive: if $f << g$ and $g << h$, then $f << h$. The proof is simply that

$$\lim \frac{f(x)}{h(x)} = \lim \frac{f(x)}{g(x)} \frac{g(x)}{h(x)} = 0.$$

If f and g are non-negative functions, then $\lim(f/g) = 0$ if and only if $\lim(g/f) = \infty$.

Example 2: By L'Hôpital's rule, $\lim \frac{x}{e^x} = \lim \frac{1}{e^x} = 0$. Now $\lim \frac{x^2}{e^x} = \lim \frac{2x}{e^x} = 0$. In the same fashion if $p(x)$ is any polynomial, then $p(x) << e^x$. If a is a positive real number, then, for an integer $n > a$, $x^a << x^n << b^n$ for $b > 1$.

Example 3: For $a > 0$,

$$\lim \frac{\ln x}{x^a} = \lim \frac{1/x}{ax^{a-1}} = \lim \frac{1}{ax^a} = 0,$$

i.e. $\ln x << x^a$.

Reasoning as in Examples 1–3, we see that if $a > 0$, $b > 1$ and $c > 1$ and $p(x)$ is a polynomial or is equal to x^a, then

$$\boxed{\log_b x << p(x) << c^x}$$

Consider functions U, V, u_i, and v_i such that $u_i << U$ and $v_i << V$ for all i and

$$\begin{aligned} f &= U + u_1 + \cdots + u_m \\ g &= V + v_1 + \cdots + v_n. \end{aligned}$$

If $U << V$ or $\lim \frac{U}{V} = a$ where $0 < a < \infty$, then, by dividing both numerator and denominator by V, we see

$$\lim \frac{f}{g} = \lim \frac{\frac{U}{V} + \frac{u_1}{V} + \ldots \frac{u_m}{V}}{1 + \frac{v_1}{V} + \ldots \frac{v_n}{V}} = \lim \frac{U}{V}.$$

If $V << U$, divide f and g by U to again get $\lim \frac{f}{g} = \lim \frac{U}{V}$.

Example 4: $\lim \dfrac{x^4 - 3x^3 + 4}{x^6 + 6x^2 + 1} = \lim \dfrac{x^4}{x^6} = 0.$

Example 5: $\lim \dfrac{e^x + x^2 + 1}{x^8 + \ln x} = \lim \dfrac{e^x}{x^8} = \infty.$

All of the statements for continuously varying functions in the material repeated above is valid also for discretely varying variables, and we continue to use the notation above for discretely varying functions. In particular, as examples of the discrete analogue of the boxed material above, we have

$$\ln n << n^3 + n + 1 \text{ or } \sqrt{n} << 2^n.$$

We would like to add two more sequences to the discrete analogue of the boxed material. For this we will use the Sandwich (Squeeze) Theorem for sequences:

Theorem: Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ eventually and $\lim a_n = \lim c_n = L$, then $\lim b_n = L$.

We define 0! and 1! both to be 1 and define $n!$ to be the product of the integers from 1 to n, for integers $n \geq 2$. The sequence {n!} grows quickly:

$$\begin{array}{ccc} 3! = 6 & 4! = 24 & 5! = 120 \\ 6! = 720 & 7! = 5040 & 8! = 40,320 \end{array}$$

Exponentials also grow quickly: $10^{10} = 10,000,000,000$, while $10! = 3,628,800$. The question is which function grows faster. L'Hôpital's Rule is not helpful because we do not know about a continuous version of the discrete $n!$. We illustrate the method for finding $\lim a^n/n!$ with the example $a = 3$: write, for $n > 4$, and $c = 9/2$,

$$0 \leq \frac{3^n}{n!} = \left(\frac{3 \cdot 3 \cdot 3}{1 \cdot 2 \cdot 3}\right)\left(\frac{3}{4} \cdots \frac{3}{n-1}\right)\frac{3}{n} \leq \frac{3c}{n}.$$

Since $\lim \frac{3c}{n} = 0$, $\lim \frac{3^n}{n!} = 0$, by the Sandwich Theorem. Thus, our "pecking order" for which functions grow faster can be extended:

$$\log_b n << p(n) << c^n << n!,$$

where b, c, p are as before.

Similar reasoning allows us to extend this list one more time:

$$0 \le \frac{n!}{n^n} = \frac{1}{n}\left(\frac{2}{n}\cdots\frac{n}{n}\right) \le \frac{1}{n}; \qquad \lim \frac{n!}{n^n} = 0;$$

$$\boxed{\log_b n << p(n) << c^n << n! << n^n}$$

We calculate some further limits that will be useful and will use the fact that if f and g are continuous functions, then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$.

1. $\lim \sqrt[n]{n} = \lim_{x\to\infty} x^{1/x} = \lim e^{\ln(x^{1/x})} = \lim e^{(\ln x)/x} = e^{\lim[(\ln x)/x]} = e^0 = 1.$

2. $\lim \sqrt[n]{a} = 1$ for each positive number a:

Since a product of real numbers less (greater) than 1 is less (greater) than 1, $\sqrt[n]{a}$ is less (greater) than 1 if and only $a = (\sqrt[n]{a})\cdots(\sqrt[n]{a})$ is less (greater) than 1.

If $a > 1$, then, since $a < n$ eventually, $1 \le \lim \sqrt[n]{a} \le \lim \sqrt[n]{n} = 1$. If $a < 1$, then $1/a > 1$, and $1 = \lim \sqrt[n]{1/a} = 1/\lim \sqrt[n]{a}$, and again $\lim \sqrt[n]{a} = 1$.

3. $\lim(1 + \frac{a}{n})^n = e^a$ for each real number a.

First we will calculate

$$\lim \ln\left(1 + \frac{a}{x}\right)^x = \lim x \ln\left(1 + \frac{a}{x}\right) = \lim \frac{\ln\left(1 + \frac{a}{x}\right)}{1/x}$$

$$= \lim \frac{\frac{-a/x^2}{1+a/x}}{-\frac{1}{x^2}} = \lim \frac{-a/x^2}{1 + a/x}\frac{-x^2}{1} = \lim \frac{a}{1 + a/x} = a.$$

$$\lim(1 + \frac{a}{n})^n = \lim(1 + \frac{a}{x})^x$$

$$= \lim e^{\ln(1+\frac{a}{x})^x} = e^{\lim \ln(1+\frac{a}{x})^x} = e^a.$$

4. $$\lim r^n = \begin{cases} \text{does not exist}, & r \leq -1 \\ 0, & 0 < |r| < 1 \\ 1, & r = 1 \\ \infty, & r > 1 \end{cases}$$

Proof: Suppose $0 < |r| < 1$, and let $a_n = |r^n|$. Then $\{a_n\}$ is a bounded monotone sequence: $0 \leq a_n \leq 1$ for all n and $a_{n+1} = |r|a_n < a_n$. Thus, $\{a_n\}$ is convergent. We denote its limit by L.

Since $a_{n+1} = |r|a_n$, by setting the limit of the left and right sides equal to each other, we obtain $L = |r|L$. Except when $L = 0$ this leads to the contradiction $L < L$. Thus, $\lim r^n = 0$, because $\lim |r_n| = 0$.

If $r > 1$, then $1/r < 1$ and $\lim r^n = \lim \frac{1}{(1/r)^n} = \infty$.

The remaining cases are obvious.

Section 10.2

Analogue Between Series and Bank Accounts

It sometimes helpful to view infinite series as bank accounts. Consider these three bank accounts

$$\sum_{n=m}^{\infty} a_n$$

Day	Amt	Bal
n	a_n	s_n
0	1	1
1	$\frac{1}{2}$	$\frac{3}{2}$
2	$\frac{1}{4}$	$\frac{7}{4}$
3	$\frac{1}{8}$	$\frac{15}{8}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{2^n}$	
		$\downarrow$
		2

Day	Amt	Bal
n	a_n	s_n
1	1	1
2	$\frac{1}{2}$	$\frac{3}{2}$
3	$\frac{1}{3}$	$\frac{11}{6}$
4	$\frac{1}{4}$	$\frac{25}{12}$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	
		$\downarrow$
		∞

Day	Amt	Bal
n	a_n	s_n
0	1	1
1	-1	0
2	1	1
3	-1	0
$\vdots$	$\vdots$	$\vdots$
n	$(-1)^n$	
		$\downarrow$
		no limit

In a typical check book, there is a column for the Date (labelled as Day above) which in mathematics is labelled as the index n.

A check book has columns marked Deposit and Withdrawal, which we combine into a single column, using positive numbers for deposits and negative numbers for withdrawals, and we use a_n as the label for a deposit or withdrawal on the nth day.

Finally, a check book has a column for the balance, and we use s_n for the balance on the nth day and call it nth partial sum since

$$s_n = a_m + \cdots + a_n = \sum_{i=m}^{n} a_i.$$

The symbol $\sum_{n=m}^{\infty} a_n$ has two meanings. First, it is the "name" of the bank account with deposits/withdrawals a_n.

The partial sums form a sequence. If the sequence has a limit (finite or infinite), the symbol $\sum_{n=m}^{\infty} a_n$ is also used to denote the limit.

Consider next the second series above. Since the deposits $a_n = 1/n$ have limit 0, intuition might lead you to believe that the limiting balance can't be very big. But, surprisingly, that is not true: by choosing n sufficiently large, we can obtain an arbitrarily large balance. For the second series, $\lim s_n = \infty$.

For the third series, the limit of the sequence $\{s_n\}$ does not exist.

Recall that, for a sequence, exactly one of these three things is true: the sequence has a finite limit, the sequence has an infinite limit, or the limit of the sequence does not exist. The sequences of balances in the three bank accounts above are examples of these three possibilities. A sequence with a finite limit we called convergent, and if the partial sums of a series is convergent, we call the series convergent. In the other two cases the sequence is called divergent, and a series is called divergent if its sequence of balances is divergent. Most of the material in the first five sections of Chapter 10 are meant to answer the question "is a given series convergent or divergent?" Only in special cases can we find an exact numerical sum of a convergent series.

An infinite sum is the discrete analogue of an improper integral for continuously varying variables:

$$\int_a^{\infty} f(x)\,dx = \lim_{M\to\infty} \int_a^{M} f(x)\,dx$$

$$\sum_{n=a}^{\infty} a_n = \lim_{M\to\infty} \sum_{n=a}^{M} a_n$$

Telescoping (Collapsing) Series

An infinite series $\sum_{n=m}^{\infty} b_n$ is said to be a *telescoping (collapsing)* series if there exists a sequence $\{a_n\}$ and a positive integer k such that $b_n = a_n - a_{n+k}$ for all $n \geq m$.

Theorem: If $\{a_n\}$ is a convergent sequence with limit L, then the telescoping series

$$\sum_{n=1}^{\infty} (a_n - a_{n+k}) = (a_1 + \cdots + a_k) - kL.$$

Proof: In one of the summations we replace i by $i-k$.

$$\begin{aligned}
s_n &= \sum_{i=1}^{n}(a_i - a_{i+k}) \\
&= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_{i+k} \\
&= \sum_{i=1}^{n} a_i - \sum_{i-k=1}^{n} a_{i-k+k} \\
&= \sum_{i=1}^{n} a_i - \sum_{i=k+1}^{n+k} a_i \\
&= (\sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i) - (\sum_{i=k+1}^{n} a_i + \sum_{i=n+1}^{n+k} a_i) \\
&= (\sum_{i=1}^{k} a_i - \sum_{i=n+1}^{n+k} a_i)
\end{aligned}$$

$$\lim s_n = \lim(\sum_{i=1}^{k} a_i - \lim \sum_{i=n+1}^{n+k} a_i)$$

$$\lim s_n = \lim(\sum_{i=1}^{k} a_i) - \sum_{i=n+1}^{n+k} \lim a_i$$

$$\lim s_n = (a_1 + \ldots + a_k) - kL.$$

Example 1: Find the value of $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

We use the partial fraction decomposition

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

$$= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} - 3 \cdot 0\right) = \frac{11}{18}.$$

Example 2: Find the value of $\sum_{n=1}^{\infty}(n^{1/n} - (n+2)^{1/(n+2)})$.

Recall that $\lim n^{1/n} = 1$.

$$\sum_{n=1}^{\infty}(n^{1/n} - (n+2)^{1/(n+2)}) = (1+\sqrt{2}) - 2 \cdot 1 = \sqrt{2} - 1.$$

Example 3: For which positive integers a is

$$\sum_{n=1}^{\infty}(a^{1/n} - a^{1/(n+2)}) \text{ an integer?}$$

Solution: This is a telescoping series with $a_n = a^{1/n}$, which converges to 1, and $k = 2$, so the sum is $a + \sqrt{a} - 2$, which is an integer if and only if a is a perfect square.

For the same sequence with $k = 3$, the series is an integer if and only if a is a perfect sixth power, since the sum is

$$a + \sqrt{a} + \sqrt[3]{a} - 3 = (a-3) + a^{1/6}(a^3 + a^2).$$

Example 4: For a real number r such that $|r| < 1$, Find $\sum_{n=1}^{\infty}(r^n - r^{n+2})$.

As two Geometric series:

$$\frac{r}{1-r} - \frac{r^3}{1-r} = \frac{r-r^3}{1-r} = \frac{r(1-r^2)}{1-r} = r(1+r).$$

As a telescoping series: $\lim r_n = 0$, so the sum is $r + r^2$.

In most texts, only the case $\lim a_n = 0$ and $k = 1$ is considered; see, for example, 10.2/Example 5 and 10.2/Exercises 5, 6 and 39-52 in the text.

Section 10.3

The following fact related to improper integrals is worth remembering, and its analogue for infinite sums is also worth remembering:

If $f(x)$ is continuous for all $x \geq a$ and $x \geq b$, then $\int_a^\infty f(x)\,dx$ and $\int_b^\infty f(x)\,dx$ are both convergent or both divergent.

The reason for this is that we have, for $a < b$,

$$\int_a^\infty f(x)\,dx = \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx.$$

Since $\int_a^b f(x)\,dx$ is finite, the left side is finite, infinite, or fails to exist if and only the second term on the right side is finite, infinite, or fails to exist, respectively.

Example: Is $\int_0^\infty e^{-x^2}\,dx$ convergent?

Solution: Let $f(x) = e^{-x^2}$ and $g(x) = e^{-x}$; $f(x)$ has no closed form integral, so we can not evaluate the given integral directly. However, for $x \geq 1$, $x^2 \geq x$, so $e^{x^2} \geq e^x$ and

$$f(x) = \frac{1}{e^{x^2}} \leq \frac{1}{e^x} = g(x),$$

and $\int_1^\infty e^{-x}\,dx = -e^{-x}\,|_1^\infty = 1/e$, so $\int_1^\infty g(x)\,dx$ is convergent, and by the Comparison Test, $\int_1^\infty f(x)\,dx$ is also convergent. Since convergence is independent of the (finite) lower limit of integration, $\int_0^\infty f(x)\,dx$ is also convergent.

Here is the discrete analogue, with a similar proof:

If a and b are integers, then $\sum_a^\infty a_n$ and $\sum_b^\infty a_n$ are both convergent or both divergent.

Example: Is $\sum_0^\infty \frac{1}{e^{n^2-n-3}}$ convergent?

Solution: Let $a_n = \frac{1}{e^{n^2-n-3}}$ and let $b_n = \frac{1}{e^n}$.
$a_n \leq b_n$ exactly when
$n^2 - n - 3 > n$ or $n^2-2n - 3 = (n+1)(n-3) > 0$.

Thus $a_n > b_n$ for $0 \leq n \leq 2$ and $a_n \leq b_n$ for $n \geq 3$. Since $\sum_3^\infty b_n$ is a geometric series with common ratio $r = 1/e$, $\sum_3^\infty b_n$ is convergent, and by the Comparison Test, $\sum_3^\infty a_n$ is convergent. Since convergence is independent of the lower limit, $\sum_0^\infty a_n$ is also convergent.

Section 10.5-6

Alternate Approach to Absolute and Conditional Convergence

For our current discussion we are going to have our "checkbook" show in the traditional way, having separate columns for the deposits, with the nth deposit denoted by d_n, and for the withdrawals, with the nth withdrawal denoted by b_n. We will consider

the series $\sum_{n=1}^{\infty} d_n$, which we will abbreviate with the symbol $\sum^{+}$, and

the series $\sum_{n=1}^{\infty} b_n$, which we will abbreviate with the symbol $\sum^{-}$.

For the alternating harmonic series, the "bank account" looks like this.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots$$

Day	Dep	Withd	Bal
n	d_n	b_n	s_n
1	1		1
2		$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{3}$		$\frac{5}{6}$
4		$\frac{1}{4}$	$\frac{7}{12}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{2n-1}$		
$n+1$		$\frac{1}{2n}$	
	$\downarrow$ ∞	$\downarrow$ ∞	$\downarrow$ $\ln 2 \approx .69$

$$d_n = \frac{1}{2n-1}.$$

$$b_n = \frac{1}{2n}.$$

$$\sum{}^{+} = \sum_{1}^{\infty} \frac{1}{2n-1} = \infty$$

$$\sum{}^{-} = \sum_{1}^{\infty} \frac{1}{2n} = \infty$$

$$\sum_{n=1}^{\infty} |a_n| = \sum{}^{+} + \sum{}^{-}$$

We will use the example above to illustrate what can happen when both $\sum^{+}$ and $\sum^{-}$ are infinite and $\lim d_n = \lim b_n =$

0. Mr. X has income which constitutes all the deposits into his account above, and pays bills which constitutes all his withdrawals. He schemes to get rich by paying his bills late as follows. Since his deposits form a divergent series, he doesn't pay any of his bills until his balance from payments received, $s_n = \sum_{n=1}^{N_1} d_n$, is over \$1000. Then he pays one bill b_1. Then he waits until the payments he is receiving brings his balance to \$2000, at which point pays he his second bill b_2. He continues these cycles of letting his deposits build his balance to n thousand dollars on the nth cycle and then paying one bill, b_n. In this fashion he becomes richer and richer, yet he still pays all his bills, just very slowly.

With the scheme above, all deposits are made and all bills eventually get paid exactly once. This is known as a *rearrangement* of the series.

A more realistic rearrangement is that many bills come in before his first deposit, and many more bills come in before his second deposit, and so forth, so that he gets deeper and deeper in debt.

With any bank account $\sum a_n$ with series $\sum^+ a_n$ of deposits and $\sum^- a_n$ of withdrawals, exactly one of three things must happen (see page 18 for a tablular summary):

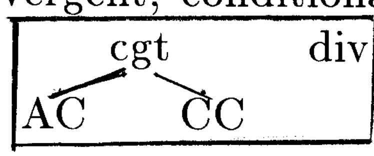
Case (1): Both $\sum^+$ and $\sum^-$ are divergent.
Case (2): Exactly one of these is divergent.
Case (3): Neither is divergent (i.e., both are convergent).

If $\sum |a_n|$ is convergent, the series $\sum a_n$ is called *absolutely convergent*. Notice that a series is absolutely convergent if and only if it is in Case (3), that is, when both $\sum^+$ and $\sum^-$ are finite. In Case (3), the original series $\sum a_n$ is convergent. That is:

An absolutely convergent series is convergent.

In Case (1) a series may also be convergent, and when this happens the series is called *conditionally convergent*. Note that in Case 1, a series is never absolutely convergent. Thus, a series is conditionally convergent if and only if it is convergent, but not absolutely convergent.

In summary, every series is either convergent or divergent, but not both. Every convergent series is either conditionally convergent or absolutely convergent, but not both. Every series is in exactly one of these categories: divergent, conditionally convergent or absolutely convergent.



Here are three observations:

1. Any time you use the Alternating Series Test a SECOND test MUST be used: Either the Alternating Series Test fails and you have to use another test or the Alternating Series Test works and you have to test $\sum |a_n|$ to see if the series AC or CC.

2. A series with only non-negative terms is absolutely convergent if it is convergent: $\sum a_n$ and $\sum |a_n|$ are the same series.

3. If $\sum a_n$ is a geometric series with common ratio r, then $\sum |a_n|$ is a geometric series with common ratio $|r|$. Thus, if a geometric series is convergent, it is absolutely convergent.

$$\sum |a_n| = \sum^+ + \sum^-$$

Case (1) Both $\sum^+ = \sum^- = \infty$: $\sum a_n$ may converge; $\sum |a_n| = \infty$

Case (2) $\sum^+ = \infty, \sum^- < \infty$: $\sum a_n = \sum |a_n| = \infty$

$\sum^+ < \infty, \sum^- = \infty$: $\sum a_n = -\infty, \sum |a_n| = \infty$

Case (3) $\sum^+, \sum^-$ both finite: $\sum a_n = \sum^+ - \sum^-$, cgt

$\sum |a_n| = \sum^+ + \sum^-$, cgt

All rearrangements give same sum.

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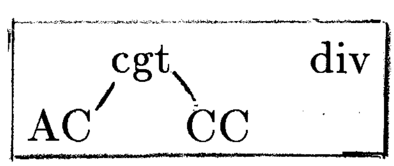
absolutely convergent $=$ $\sum |a_n|$ cgt $\Rightarrow \sum a_n$ cgt

conditionally convergent $=$ $\sum a_n$ cgt, but $\sum |a_n|$ div



Procedure to decide AC, CC, or Div:

(1) Test $\sum a_n$; if it is Div, you're done.

(2) Otherwise ALSO test $\sum |a_n|$:

If $\sum |a_n|$ cgt, answer is AC.

If $\sum |a_n|$ div, answer is CC.

The Ratio and Root Test Limits

Suppose that a_n is a sequence of real numbers. It is a fact, which we will not prove, that if $\lim |a_{n+1}/a_n|$ exists then so does $\lim \sqrt[n]{|a_n|}$ and the two limits are equal. It can happen that $\lim \sqrt[n]{|a_n|}$ may exist, but $\lim |a_{n+1}/a_n|$ does not, and it can happen that neither limit exists.

Example 1 (=Example 3 from text Section 10.5, modified to be an alternating series):

$\lim \sqrt[n]{|a_n|}$ exists, but $\lim |a_{n+1}/a_n|$ does not exist. This also provides an example of an alternating series $\sum a_n$ for which the Alternating Series Test fails, yet the series is absolutely convergent.

$$\sum_1^\infty a_n = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{1}{16} + \frac{5}{32} - \frac{1}{64} + \cdots + \frac{n}{2^n} - \frac{1}{2^n} \pm \cdots$$

$$= (\frac{1}{2} - \frac{1}{4}) + (\frac{3}{8} - \frac{1}{16}) + (\frac{5}{32} - \frac{1}{64}) + \cdots + (\frac{n}{2^n} - \frac{1}{2^n}) + \cdots .$$

Example 2: Neither $\lim \sqrt[n]{|a_n|}$ nor $\lim |a_{n+1}/a_n|$ exist. In this example the Alternating Series Test fails and the series $\sum a_n$ is divergent.

$$\sum_1^\infty a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{8} \pm \cdots + \frac{1}{n} - \frac{1}{2^n} \pm \cdots$$

$$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{8}) + \cdots + (\frac{1}{n} - \frac{1}{2^n}) + \cdots .$$

If we replace the terms $\frac{1}{n}$ by $\frac{1}{n^2}$ in this example, we obtain an example of an alternating absolutely convergent series for which neither $\lim \sqrt[n]{|a_n|}$ nor $\lim |a_{n+1}/a_n|$ exist and the Alternating Series Test fails. The series is absolutely convergent because $\sum^+ = \sum \frac{1}{n^2}$ and $\sum^- = \sum \frac{1}{2^n}$ are finite.