Math 205 Test 2
November 3, 2016

Name: ______________________________Solutions______________________________

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Calculators are NOT allowed. Also, you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

How spooky it would be to get 120 on this exam?? (In a good way)

1. (a) (15 points) Let $f(x) = 2 - \frac{5}{x}$. Use the limit definition of the derivative to find $f'(x)$. **No credit will be given for any other method!**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2 - \frac{5}{x+h} - \left(2 - \frac{5}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{5}{x+h} + \frac{5}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h\to 0} \frac{-5x + 5(x+h)}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-5\cancel{x} + 5\cancel{x} + 5\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h\to 0} \frac{5}{x(x+h)}$$

$$\Rightarrow \boxed{f'(x) = \frac{5}{x^2}}$$

(b) (5 points) Using your answer to part (a), compute the equation of the tangent line to $f(x)$ at the point where $x = 1$. **Write your line in $y = mx + b$ form.**

$x_1 = 1$

$y_1 = 2 - \frac{5}{1} = -3$

$m = f'(1) = \frac{5}{(1)^2} = 5$

Using $y - y_1 = m(x - x_1)$

We get $\rightarrow$ $y - (-3) = 5(x - 1)$

$\Rightarrow y + 3 = 5x - 5$

$\Rightarrow \boxed{y = 5x - 8}$

2. Find $\frac{dy}{dx} = y'$ for the following. Simplify your answers. (4 points each)

(a) $y = \frac{5x^3+3x^2}{2x}$

$$= \frac{5}{2}x^2 + \frac{3}{2}x$$

$$\Rightarrow \boxed{y' = 5x + \frac{3}{2}}$$

(b) $y = 2\sqrt{x} + \frac{5}{\sqrt[3]{x}} - \ln(x^2+1)^3$

$$= 2x^{1/2} + 5x^{-1/3} - 3\ln(x^2+1)$$

$$\Rightarrow \boxed{y' = x^{-1/2} - \frac{5}{3}x^{-4/3} - 3 \cdot \frac{2x}{x^2+1}}$$

$$\text{OR } \boxed{y' = x^{-1/2} - \frac{5}{3}x^{-4/3} - \frac{6x}{x^2+1}}$$

(c) $y = \frac{x^6}{4+x^6}$

$$y' = \frac{(4+x^6)(6x^5) - x^6(6x^5)}{(4+x^6)^2}$$

$$= \frac{6x^5(4+\cancel{x^6}-\cancel{x^6})}{(4+x^6)^2}$$

$$\boxed{y' = \frac{24x^5}{(4+x^6)^2}}$$

(d) $y = e^{x^2} + x^{x^2}$

One method

First: $y_2 = x^{x^2}$

$$\Rightarrow \ln y_2 = \ln x^{x^2} = x^2 \ln x$$

$$\Rightarrow \frac{y_2'}{y_2} = 2x\ln x + x^2\left(\frac{1}{x}\right)$$

$$\Rightarrow y_2' = y_2(2x\ln x + x) = x^{x^2}(2x\ln x + x)$$

$$\Rightarrow \boxed{y' = 2xe^{x^2} + x^{x^2}(2x\ln x + x)}$$

Another method

$$y = e^{x^2} + e^{\ln x^{x^2}}$$

$$= e^{x^2} + e^{x^2 \ln x}$$

$$\Rightarrow y' = 2xe^{x^2} + \left(2x\ln x + x^2 \cdot \frac{1}{x}\right)e^{x^2 \ln x}$$

$$\boxed{y' = 2xe^{x^2} + x^{x^2}(2x\ln x + x)}$$

(e) $x^2y^3 + 2x + 3y = 5x + 12$

$\Rightarrow 2xy^3 + x^2 \cdot 3y^2 y' + 2 + 3y' = 5$

$\Rightarrow y'(3x^2y^2+3) = 5 - 2xy^3 - 2 = 3 - 2xy^3$

$\Rightarrow \boxed{y' = \dfrac{3-2xy^3}{3x^2y^2+3}}$

3. (5 points each part) A bunch of angry calculus students (allegedly) throw Jhevon off a cliff. Jhevon's position above the ground at time t seconds is given by $s(t) = -16t^2 + 16t + 96$.

(a) Find functions that describe Jhevon's velocity and acceleration at time t.

$\boxed{v(t) = -32t + 16} \longrightarrow$ velocity $= s'(t)$.

$\boxed{a(t) = -32} \longrightarrow$ acceleration $= v'(t) = a'(t)$.

(b) When will Jhevon hit the ground and the nightmare end for his students?

Jhevon hits ground when position $= 0$

$\Rightarrow -16t^2 + 16t + 96 = 0$

$\Rightarrow -16(t^2 - t - 6) = 0$

$\Rightarrow -16(t-3)(t+2) = 0$

$\Rightarrow t = 3, \quad \cancel{t = -2}$

reject!

Jhevon will hit the ground at $\boxed{t=3}$ seconds

(c) What is the highest height Jhevon attains?

Highest height occurs when $v(t) = 0$

$\Rightarrow -32t + 16 = 0$

$\Rightarrow t = 1/2$

$\Rightarrow$ Highest height $= s(\frac{1}{2})$

$= -16(\frac{1}{2})^2 + 16(\frac{1}{2}) + 96$

$= -4 + 8 + 96$

$= \boxed{100}$ units

(d) With what velocity will Jhevon hit the ground? This number shall be commemorated with fond memories.

He hits the ground when $t=3$.

$\Rightarrow$ We want $v(3) = -32(3)+16$

$= \boxed{-80}$ units/sec

4. (i) (2 points each) State the following rules precisely:

(a) The power rule for derivatives: $\frac{d}{dx} x^n = nx^{n-1}$

(b) The chain rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

(c) The quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

(d) The product rule:

$$\frac{d}{dx}(f \cdot g) = f'g + fg'$$

(f) The rule to differentiate a general exponential with base a and power u. $\frac{d}{dx} a^u = u' a^u \ln a$

(g) The rule to differentiate the natural logarithm of a function u.

$$\frac{d}{dx} \ln u = \frac{u'}{u}$$

(ii) (8 points) Use linear approximation to approximate $\sqrt[3]{26.9}$. You may leave your answer as a sum of fractions.

Use $f(x) \approx f(a) + f'(a)(x-a)$

with $x = 26.9$, $a = 27$

$f(x) = \sqrt[3]{x}$

$\Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x}^2}$

$\Rightarrow f(a) = f(27) = 3$

$\Rightarrow f'(a) = f'(27) = \frac{1}{27}$

$$\Rightarrow \sqrt[3]{26.9} \approx 3 + \frac{1}{27}(26.9 - 27)$$

$$= 3 + \frac{1}{27}\left(-\frac{1}{10}\right)$$

$$= 3 - \frac{1}{270}$$

$$= \boxed{\frac{809}{270}}$$

5. (a) We return to our story, where our hero, Jhevon, is trying to get his hotdog business to be the very best, like no one ever was. The cost $C(x)$, in dollars, of producing x hotdogs is given by

$$C(x) = 50 - 20x + 2x^2$$

Assuming Jhevon will sell only specialty hotdogs at \$5/hotdog, answer the following:

i. (6 points) What is Jhevon's revenue function, $R(x)$, and profit function, $P(x)$?

$\boxed{R(x) = 5x}$ $\quad P(x) = R(x) - C(x) \Rightarrow \boxed{P(x) = -50 + 25x - 2x^2}$

ii. (4 points) Find the marginal cost and marginal revenue functions.

$\boxed{C'(x) = -20 + 4x} \longrightarrow$ marginal cost.

$\boxed{R'(x) = 5} \longrightarrow$ Marginal Revenue.

iii. (4 points) Assume Jhevon made 6 hotdogs, use the marginal cost to approximate how much more it would cost him to make the seventh.

$C'(6) = -20 + 4(6) = 4$

$\Rightarrow$ $\boxed{\text{It would cost him \$4 more.}}$

(b) Compute the following limits (2 points each):

i. $\lim_{x\to 1} \frac{2+x+x^2}{x^2-4} = \frac{2+(1)+(1)^2}{(1)^2-4}$

$= \boxed{-\frac{4}{3}}$

ii. $\lim_{x\to -\infty} \frac{2-3x^3+2x}{5-4x+2x^3} = \boxed{-\frac{3}{2}}$

iii. $\lim_{x\to -3} \frac{x^2+4x+3}{9-x^2} = \lim_{x\to -3} \frac{(\cancel{x+3})(x+1)}{(3-x)(\cancel{3+x})}$

$= \frac{-2}{6}$

$= \boxed{-\frac{1}{3}}$

Bonus Problems: (You must complete all problems in the actual test to be eligible).

1. Jhevon, with the aid of his accomplice, rob a bank, making off with \$3,000,000. Jhevon puts the money in a Swiss bank account that earns 5% annual interest compounded continuously.
   (a) (2 points) Write down a differential equation, with initial condition, that describes Jhevon's bank balance, P, at time t years after account opening.

   $$P' = 0.05P, \quad P(0) = 3000000$$

   (b) (3 points) Find the formula for the function $P(t)$.

   $$P = 3000000\, e^{0.05t}$$

2. (8 points) The concentration of a drug in a patient's bloodstream t hours after it is taken is given by

   $$C(t) = \frac{0.016t}{(t+2)^2}\, mg/cm^3.$$

   Find the maximum concentration of the drug and the time at which it occurs.

$$C'(t) = \frac{(t+2)^2 \cdot 0.016 - 0.016t \cdot 2(t+2)}{(t+2)^4}$$

$$= \frac{0.016\cancel{(t+2)}\,[t+2-2t]}{(t+2)^{\cancel{4}\,3}}$$

$$= \frac{0.016(2-t)}{(t+2)^3}$$

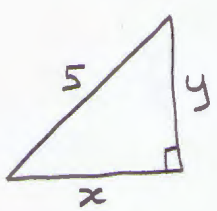
$\Rightarrow$ crit pts: $t = 2$, $\cancel{t = -2}$ reject!

Test: C': 0, ① (+), 2, ③ (−)

$$\boxed{\text{Max Conc} = C(2) = \frac{1}{500}}$$

$\boxed{\text{Time for max conc} = 2}$ hours

3. (7 points) A 5 foot ladder leans against a vertical wall. Batman pushes the foot of the ladder towards the wall at a rate of 2 ft/sec. At what rate is top of the ladder moving along the wall when the foot of the ladder is 3 feet from the wall? Include a sketch in your answer.



Know: $\frac{dx}{dt} = -2$

Want: $\frac{dy}{dt}$ when $x = 3$ $\downarrow$ $y = 4$ since 3-4-5 △

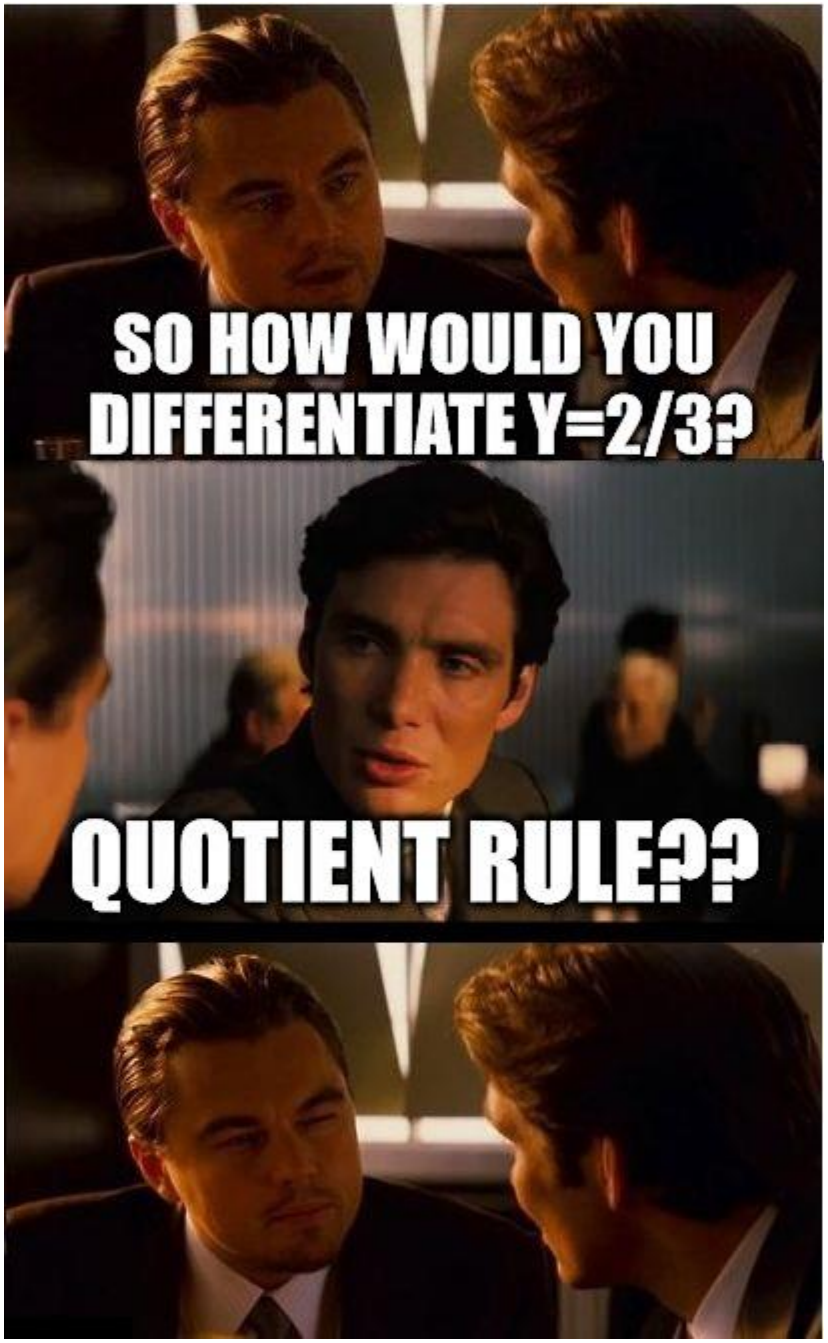
$$x^2 + y^2 = 5^2$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y}$$

$$= \frac{-(3)(-2)}{4}$$

$$\boxed{\frac{dy}{dt} = \frac{3}{2}}\ \text{ft/sec}$$


SO HOW WOULD YOU
DIFFERENTIATE Y=2/3?
QUOTIENT RULE??