Math 205 Test 2
November 2, 2016

Name: ____________________________Solutions____________________________

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional. And they will only be counted if all other problems are attempted.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers.
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. Calculators are NOT allowed. Also, you are NOT allowed to use notes, or other aids—including, but not limited to, divine intervention/inspiration, the internet, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Other than that, have fun and good luck!

How spooky it would be to get 120 on this exam?? (In a good way)

1. (a) (15 points) Let $f(x) = 2 - \frac{5}{x}$. Use the limit definition of the derivative to find $f'(x)$. **No credit will be given for any other method!**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2 - \frac{5}{x+h} - \left(2 - \frac{5}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-\frac{5}{x+h} + \frac{5}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$= \lim_{h \to 0} \frac{-5x + 5(x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{\cancel{-5x} + \cancel{5x} + 5\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h \to 0} \frac{5}{x(x+h)}$$

$$\Rightarrow \boxed{f'(x) = \frac{5}{x^2}}$$

(b) (5 points) Using your answer to part (a), compute the equation of the tangent line to $f(x)$ at the point where $x = 1$. **Write your line in $y = mx + b$ form.**

$x_1 = 1$

$y_1 = 2 - \frac{5}{1} = -3$

$m = f'(1) = \frac{5}{(1)^2} = 5$

Using $y - y_1 = m(x - x_1)$

We get

$y - (-3) = 5(x - 1)$

$\Rightarrow y + 3 = 5x - 5$

$\Rightarrow \boxed{y = 5x - 8}$

2. Find $\frac{dy}{dx} = y'$ for the following. Simplify your answers. (4 points each)

(a) $y = \frac{5x^3+3x^2}{2x}$

$$= \frac{5}{2}x^2 + \frac{3}{2}x$$

$$\Rightarrow \boxed{y' = 5x + \frac{3}{2}}$$

(b) $y = 2\sqrt{x} + \frac{5}{\sqrt[3]{x}} - \ln(x^2+1)^3$

$$= 2x^{1/2} + 5x^{-1/3} - 3\ln(x^2+1)$$

$$\Rightarrow \boxed{y' = x^{-1/2} - \frac{5}{3}x^{-4/3} - 3\cdot\frac{2x}{x^2+1}}$$

OR $\boxed{y' = x^{-1/2} - \frac{5}{3}x^{-4/3} - \frac{6x}{x^2+1}}$

(c) $y = \frac{x^6}{4+x^6}$

$$y' = \frac{(4+x^6)(6x^5) - x^6(6x^5)}{(4+x^6)^2}$$

$$= \frac{6x^5(4+\cancel{x^6}-\cancel{x^6})}{(4+x^6)^2}$$

$$\boxed{y' = \frac{24x^5}{(4+x^6)^2}}$$

(d) $y = e^{x^2} + x^{x^2}$

One method

First: $y_2 = x^{x^2}$

$$\Rightarrow \ln y_2 = \ln x^{x^2} = x^2 \ln x$$

$$\Rightarrow \frac{y_2'}{y_2} = 2x\ln x + x^2\left(\frac{1}{x}\right)$$

$$\Rightarrow y_2' = y_2(2x\ln x + x) = x^{x^2}(2x\ln x + x)$$

$$\Rightarrow \boxed{y' = 2xe^{x^2} + x^{x^2}(2x\ln x + x)}$$

Another method

$$y = e^{x^2} + e^{\ln x^{x^2}}$$

$$= e^{x^2} + e^{x^2 \ln x}$$

$$\Rightarrow y' = 2xe^{x^2} + \left(2x\ln x + x^2\cdot\frac{1}{x}\right)e^{x^2\ln x}$$

$$\boxed{y' = 2xe^{x^2} + x^{x^2}(2x\ln x + x)}$$

(e) $x^2y^3 + 2x + 3y = 5x + 12$

$\Rightarrow 2xy^3 + x^2 \cdot 3y^2 y' + 2 + 3y' = 5$

$\Rightarrow y'(3x^2y^2 + 3) = 5 - 2xy^3 - 2 = 3 - 2xy^3$

$\Rightarrow \boxed{y' = \dfrac{3 - 2xy^3}{3x^2y^2 + 3}}$

3. (5 points each part) A bunch of angry calculus students (allegedly) throw Jhevon off a cliff. Jhevon's position above the ground at time t seconds is given by $s(t) = -16t^2 + 16t + 96$.

(a) Find functions that describe Jhevon's velocity and acceleration at time t.

$\boxed{v(t) = -32t + 16} \longrightarrow$ velocity $= s'(t)$.

$\boxed{a(t) = -32} \longrightarrow$ acceleration $= v'(t) = a'(t)$.

(b) When will Jhevon hit the ground and the nightmare end for his students?

Jhevon hits ground when position $= 0$

$\Rightarrow -16t^2 + 16t + 96 = 0$

$\Rightarrow -16(t^2 - t - 6) = 0$

$\Rightarrow -16(t-3)(t+2) = 0$

$\Rightarrow t = 3, \; \cancel{t = -2}$

reject!

Jhevon will hit the ground at $\boxed{t = 3}$ seconds

(c) What is the highest height Jhevon attains?

Highest height occurs when $v(t) = 0$

$\Rightarrow -32t + 16 = 0$

$\Rightarrow t = 1/2$

$\Rightarrow$ Highest height $= s(\tfrac{1}{2})$

$= -16(\tfrac{1}{2})^2 + 16(\tfrac{1}{2}) + 96$

$= -4 + 8 + 96$

$= \boxed{100}$ units

(d) With what velocity will Jhevon hit the ground? This number shall be commemorated with fond memories.

He hits the ground when $t=3$.

$\Rightarrow$ We want $v(3) = -32(3)+16$

$= \boxed{-80}$ units/sec

4. (i) (2 points each) State the following rules precisely:

(a) The power rule for derivatives: $\frac{d}{dx}x^n = nx^{n-1}$

(b) The chain rule:

$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$$

(c) The quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

(d) The product rule:

$$\frac{d}{dx}(f\cdot g) = f'g + fg'$$

(f) The rule to differentiate a general exponential with base a and power u. $\frac{d}{dx}a^u = u'a^u \ln a$

(g) The rule to differentiate the natural logarithm of a function u.

$$\frac{d}{dx}\ln u = \frac{u'}{u}$$

(ii) (8 points) Use linear approximation to approximate $\sqrt[3]{26.9}$. You may leave your answer as a sum of fractions.

Use $f(x) \approx f(a) + f'(a)(x-a)$

with $x = 26.9$, $a = 27$

$f(x) = \sqrt[3]{x}$

$\Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x}^2}$

$\Rightarrow f(a) = f(27) = 3$

$\Rightarrow f'(a) = f'(27) = \frac{1}{27}$

$$\Rightarrow \sqrt[3]{26.9} \approx 3 + \frac{1}{27}(26.9-27)$$

$$= 3 + \frac{1}{27}\left(-\frac{1}{10}\right)$$

$$= 3 - \frac{1}{270}$$

$$= \boxed{\frac{809}{270}}$$

5. (a) We return to our story, where our hero, Jhevon, is trying to get his hotdog business to be the very best, like no one ever was. The cost $C(x)$, in dollars, of producing x hotdogs is given by

$$C(x) = 50 - 20x + 2x^2$$

Assuming Jhevon will sell only specialty hotdogs at $5/hotdog, answer the following:

i. (6 points) What is Jhevon's revenue function, $R(x)$, and profit function, $P(x)$?

$\boxed{R(x) = 5x}$ $\quad P(x) = R(x) - C(x) \Rightarrow \boxed{P(x) = -50 + 25x - 2x^2}$

ii. (4 points) Find the marginal cost and marginal revenue functions.

$\boxed{C'(x) = -20 + 4x} \longrightarrow$ marginal cost.

$\boxed{R'(x) = 5} \longrightarrow$ Marginal Revenue.

iii. (4 points) Assume Jhevon made 6 hotdogs, use the marginal cost to approximate how much more it would cost him to make the seventh.

$C'(6) = -20 + 4(6) = 4$

$\Rightarrow$ $\boxed{\text{It would cost him \$4 more.}}$

(b) Compute the following limits (2 points each):

i. $\lim_{x\to 1} \frac{2+x+x^2}{x^2-4} = \frac{2+(1)+(1)^2}{(1)^2-4}$

$= \boxed{-\frac{4}{3}}$

ii. $\lim_{x\to -\infty} \frac{2-3x^3+2x}{5-4x+2x^3} = \boxed{-\frac{3}{2}}$

iii. $\lim_{x\to -3} \frac{x^2+4x+3}{9-x^2} = \lim_{x\to -3} \frac{\cancel{(x+3)}(x+1)}{(3-x)\cancel{(3+x)}}$

$= \frac{-2}{6}$

$= \boxed{-\frac{1}{3}}$

Bonus Problems: (You must complete all problems in the actual test to be eligible).

1. Jhevon, with the aid of his accomplice, rob a bank, making off with \$3,000,000. Jhevon puts the money in a Swiss bank account that earns 5% annual interest compounded continuously.
   (a) (2 points) Write down a differential equation, with initial condition, that describes Jhevon's bank balance, P, at time t years after account opening.

$$P' = 0.05P, \quad P(0) = 3000000$$

   (b) (3 points) Find the formula for the function $P(t)$.

$$P = 3000000\, e^{0.05t}$$

2. (8 points) The concentration of a drug in a patient's bloodstream t hours after it is taken is given by

$$C(t) = \frac{0.016t}{(t+2)^2}\, mg/cm^3.$$

Find the maximum concentration of the drug and the time at which it occurs.

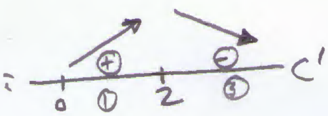
$$C'(t) = \frac{(t+2)^2 \cdot 0.016 - 0.016t \cdot 2(t+2)}{(t+2)^4}$$

$$= \frac{0.016\cancel{(t+2)}[t+2-2t]}{(t+2)^{\cancel{4}\,3}}$$

$$= \frac{0.016(2-t)}{(t+2)^3}$$

$\Rightarrow$ crit pts: $t=2$, $\cancel{t=-2}$ reject!

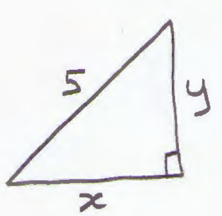
Test:



$$\boxed{\text{Max Conc} = C(2) = \frac{1}{500}}$$

$$\boxed{\text{Time for max conc} = 2} \text{ hours}$$

3. (7 points) A 5 foot ladder leans against a vertical wall. Batman pushes the foot of the ladder towards the wall at a rate of 2 ft/sec. At what rate is top of the ladder moving along the wall when the foot of the ladder is 3 feet from the wall? Include a sketch in your answer.



Know: $\frac{dx}{dt} = -2$

Want: $\frac{dy}{dt}$ when $x = 3$ $\downarrow$ $y = 4$ since 3-4-5 $\triangle$

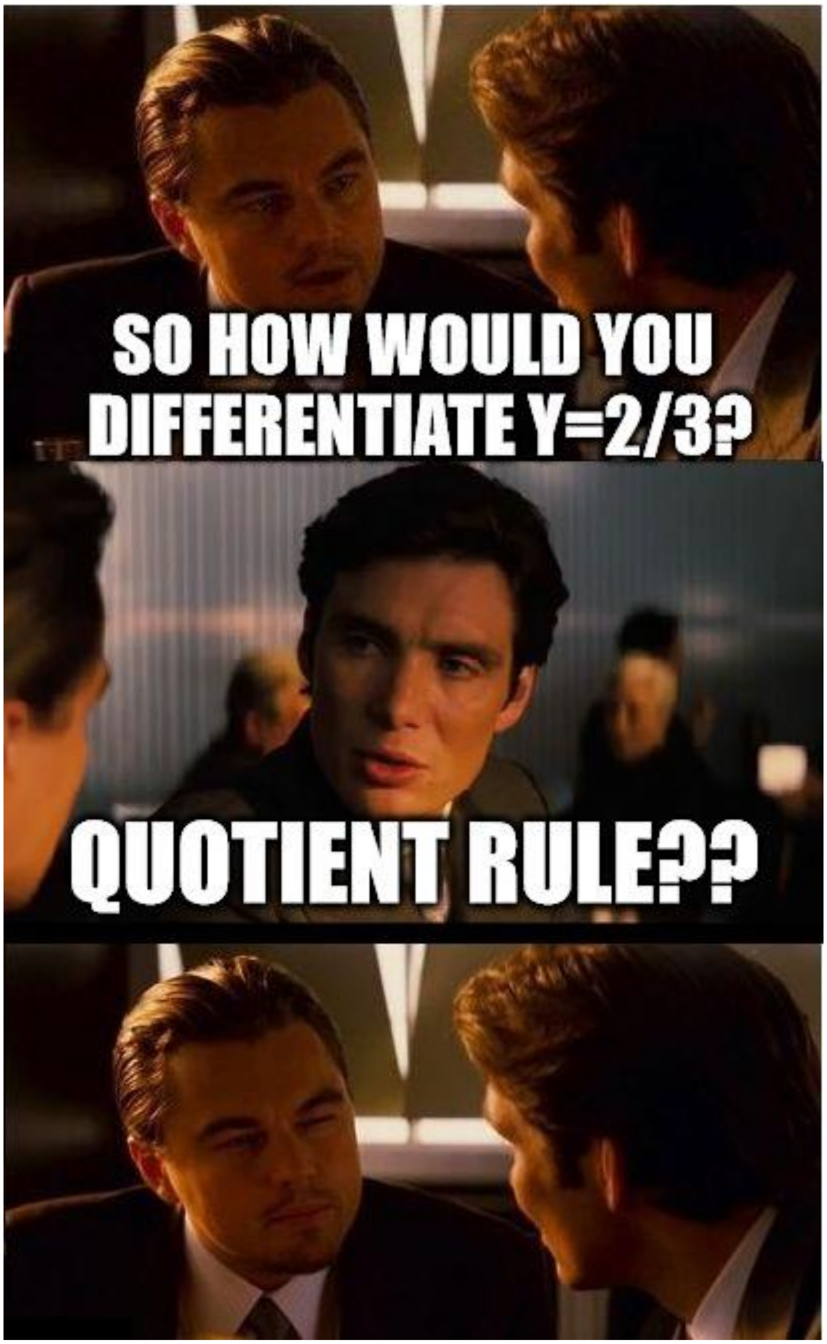
$$x^2 + y^2 = 5^2$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y}$$

$$= \frac{-(3)(-2)}{4}$$

$$\boxed{\frac{dy}{dt} = \frac{3}{2}} \text{ ft/sec}$$


SO HOW WOULD YOU
DIFFERENTIATE Y=2/3?
QUOTIENT RULE??

Math 205 Test 2 Grading Guidelines

Problem 1 – 15 points for part a, 5 points for part b.

Part (a): Student needed to find the derivative using the definition of the derivative. The student would receive zero points if they used any other method. The student may check their work (and should) using a shorter method (in this case, the power rule), but the student would receive no credit for checking.
If the student stated the definition and or set up line 2 correctly, including the correct use of parentheses, the student will receive 5 out of the 15 points. 8 points is given for the rest of the working out, and 2 points for writing the final answer $f'(x) = \frac{5}{x^2}$.

At this point, the student is not expected to make any crazy algebraic mistakes…

Part (b):
1 point for finding the corresponding y-value.
1 point for finding the slope, m, using the answer to part (a).
1 point for plugging things into any form of the equation of a line.
2 points to simplify to the final answer $y = 5x - 8$. This form is specifically required, so the student will lose these 2 points even if they have the right answer in a different form.

Problem 2 – 4 points each part.

Part (a): Best method is to simplify and use power rule. It is not likely to make a mistake by doing this, so full points should be ok once they get to the (correct) final answer. Some student may try to use the quotient rule (why?? See the meme on the previous page), in this case, any mistakes made will be punished heavily. A minimum of 2 points will be deducted. Especially if the answer is not simplified. The comment "Quotient rule is overkill!!!" or "Overkill" will either be explicitly written or implied. An acceptable answer would be $y' = \frac{10x+3}{2}$, which you may get using the quotient rule.

If quotient rule is used, 2 points for correct set up, 1 point to calculate, 1 point for final answer.

Part (b):
2 points to rewrite in a convenient form.
1 point to differentiate correctly.
1 point to simplify to get the correct final answer.

Part (c):
2 points to set up line 1, correctly.
1 point to calculate.
1 point to simplify to the final answer.

The student may rewrite the problem as $y = x^6(4 + x^6)^{-1}$ and proceed to use the product rule with the chain rule. This is OK, as long as it is done correctly. But the final answer must be simplified to the form $y' = \frac{24x^5}{(4+x^6)^2}$ or $y' = 24x^5(4 + x)^{-2}$.

Part (d): Student can use either method presented here, most would probably use method 1—log differentiation. In this case, the two terms must be split up(*).

1 point for the correct final answer.
3 points for correct set up and simplifying.

(*) If the student tried log differentiation and did not split things up, but erroneously thought that logs distribute across sums, that is, the right side became $\ln e^{x^2} + \ln x^{x^2}$, then the student will receive **zero points** for this part.

Part (e): Implicit differentiation is needed here.
1 point to differentiate correctly, applying all the necessary rules, like the product rule, etc.
2 points for simplifying.
1 point to get the correct final answer.

If the student messes up the first line, such as not apply the product rule, **zero points** will be given for this part.

Problem 3 – 5 points each part. Units are not important here.
Part (a):
3 points for finding velocity.
2 points for finding acceleration.
ALL OR NOTHING. A student really shouldn't mess this up at this point.

Part (b):
2 points to figure out the right equation to use, that is, $-16t^2 + 16t + 96 = 0$.
2 points to solve.
1 point for correct final answer. The answer $\boxed{t = 3}$ needs to be boxed or otherwise indicated as the answer. If the answer is left as $t = 3 \; or \; t = -2$, then this point will be lost.

Part (c):
2 points for knowing to use the velocity equation to find time first.
1 point for finding $t = 1/2$
1 point for knowing to plug into $s(t)$
1 point to calculate and get the correct final answer of $\max height = 100$

Part (d):
2 points for knowing to use $t = 3$ (or the answer from part (b), even if the student got part b wrong, they will get the points for using that answer here—unless the answer is *really* wrong, like negative time or something).
2 points for knowing to plug this into $v(t)$
1 point to simplify to get the correct answer, $v(3) = -80$.

Problem 4 – 2 points each part for part (i), 8 points for part (ii).
Part (i) – parts (a) through (g):
2 points each part, all or nothing. The rules have been asked on many quizzes, and reviewed at the beginning of many classes—there is NO excuse for not knowing them at this point. Even a slight mistake or omission will cause all the points lost for the parts in part (i).

Part (ii):
1 points for knowing the linear approximation formula.
2 points for finding the correct $f(x)$.
1 point for differentiating correctly.
1 point to find $f(a)$
1 point to find $f'(a)$
1 point to plug into the linear approximation formula.
1 point to simplify to the final answer of $\frac{809}{270}$. The answer is required to be in fraction form, or a sum of fractions, so leaving the answer as $3 - \frac{1}{270}$ is okay.

Problem 5 – points vary per part.
Part (a)(i):
2 points to find $R(x)$.
4 points to find the correct $P(x)$.

In finding $P(x)$, a common error may be not using parentheses, and hence not subtracting the terms of the cost function correctly. A minimum of 2 points will be deducted for this.

Part(a)(ii):
2 points to find $C'(x)$
2 points to find $R'(x)$
All or nothing. Mistakes should not be made at this point. If the student got $R(x)$ wrong for the first part, then they will get the full points for differentiating whatever their answer was correctly.

Part (a)(iii):
3 points to know that they want to use $C'(6)$ and plug in 6 into $C'(x)$.
1 point for the correct final answer of \$4.

Part (b)(i):
1 point for plugging in $x = 1$.
1 point for correct final answer.

Part (b)(ii):
2 points, all or nothing, for the answer. Student should know to just look at the highest powers here. The highest power in the numerator and denominator are the same, so the answer is the ratio of the leading coefficients.

Part (b)(iii):

1 point to factor correctly and simplify.
1 point to get the correct final answer of $-\frac{1}{3}$.

Bonus

Problem 1 part a: All or nothing, this is just plugging in numbers into formulas that should be memorized.

Problem 1 part b: All or nothing. For the same reason as problem 1a.

Problem 2:
2 points for finding the derivative.
2 points for choosing the correct critical point to test.
2 points for testing the critical point.
1 point to find the max concentration by plugging in that critical point into the original function.
1 point for stating that the critical point found is the time needed to attain the max concentration.

Problem 3:
2 points for a correctly labeled diagram. The letters used may be different from those used in the solutions, but variables should appear in the same positions, and constants in the same positions as in the solutions.
1 points for setting up the right equation $x^2 + y^2 = 5^2$ (could be with different variables).
2 points to differentiate implicitly correctly.
1 point for figuring out all the correct numbers to plug in and then plugging them in in the correct positions.
1 point to get the final answer of $\frac{3}{2}\ ft/\sec$ (it's OK if units are not used).

Required knowledge
For this test, students needed to know how to differentiate using all the techniques taught in class and to remember all the derivative, log and exponential rules, as well as algebra! Students also needed to know how to find the derivative using the definition of the derivative and find the equation of the tangent line to a function at a given point. Students needed to understand and apply the physics of motion, and knowing, for example, how derivatives connect position, velocity and acceleration. Students needed to know the linear approximation formula, and know how to apply it to approximate a radical. A knowledge of marginal analysis was also required, as well as how to set up simple functions used in economics, such as revenue and profit. Students also had to be able to compute limits, using various techniques shown in class.

Optional knowledge
To gain bonus points, students had to know the formulas for exponential growth and how to figure out where to plug numbers into these formulas. Students would also need to know how to find maximum points—this was discussed in class when discussing the purpose of a function and its first and second derivative. Students also needed to know how to set up and solve a simple related rates problem.