

Name: JHEVON SMITH

Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems in the actual test. Bonus problems are, of course, optional.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean!  $x^2$  and  $x2$  are not the same thing, for example, and I will grade accordingly.
10. Do NOT commit any of the blasphemies or mistakes I mentioned in the syllabus. I will actually mete out punishment in the way I said I would. I wasn't kidding.
11. Other than that, have fun and good luck!

Remember: Don't be like John.

**DERP: SO JOHN, HOW DO I SOLVE  
THIS PROBLEM?  
JOHN: GET A GUN**



**DON'T BE LIKE JOHN. REASON  
THROUGH YOUR PROBLEMS**

1. (5 points each) Find the limit if it exists (justify your answer!), OR if the limit does not exist, explain why.

$$(a) \lim_{(x,y) \rightarrow (0,0)} \frac{y \sin x}{x^2 + y^2} = L$$

along  $y=0$ ,  $L=0$

$$\begin{aligned} \text{along } x=y, L &= \lim_{x \rightarrow 0} \frac{x \sin x}{2x^2} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2x} \\ &= \frac{1}{2} \end{aligned}$$

$\therefore L$  D.N.E.

$$(b) \lim_{(x,y) \rightarrow (0,0)} \frac{2y^2}{2x^2 - y^2} = L$$

along  $y=0$ ,  $L=0$

$$\text{along } x=0, L = \lim_{y \rightarrow 0} \frac{2y^2}{-y^2} = -2$$

$\therefore L$  D.N.E.

$$(c) \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = L$$

Note that  $\left| \frac{xy^2}{x^2 + y^2} - 0 \right|$

$$= \left| \frac{xy^2}{x^2 + y^2} \right|$$

$$\leq \left| \frac{xy^2}{y^2} \right|$$

$$g(x,y) \leftarrow = |x| \rightarrow 0 \text{ as } (x,y) \rightarrow (0,0)$$

$\therefore L=0$  by the squeeze theorem!

$$(d) \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 \sin y}{2x^2 + y^2} = L$$

Note that  $\left| \frac{2x^2 \sin y}{2x^2 + y^2} - 0 \right|$

$$= \left| \frac{2x^2 \sin y}{2x^2 + y^2} \right|$$

$$\leq \left| \frac{2x^2 \sin y}{2x^2} \right|$$

$$g(x,y) \leftarrow = |\sin y| \rightarrow 0 \text{ as } (x,y) \rightarrow (0,0)$$

$\therefore L=0$  by the squeeze theorem!

2. (20 points) Find and classify all critical points of  $f(x, y) = y^2 + x^2y + x^2 - 2y$ .

$$f_x = 2xy + 2x$$

$$f_y = 2y + x^2 - 2$$

$$f_{xx} = 2y + 2$$

$$f_{yy} = 2$$

$$f_{xy} = 2x$$

For crit. pts:

$$2xy + 2x = 0$$

$$\text{and } 2y + x^2 - 2 = 0$$

$$\Rightarrow 2x(y+1) = 0$$

$$\Rightarrow x=0 \text{ or } y=-1$$

when  $x=0$

$$2y - 2 = 0$$

$$\Rightarrow y=1$$

$$\boxed{(0, 1)}$$

when  $y=-1$

$$-2 + x^2 - 2 = 0$$

$$\Rightarrow x = \pm 2$$

$$\boxed{(2, -1)}$$

$$\text{and } \boxed{(-2, -1)}$$

Test (0, 1)

$$D = 4(2) - 0^2 > 0$$

and  $f_{xx} > 0$

$\Rightarrow$   $(0, 1)$  is a minimum!

Test (2, -1)

$$D = 0 - 4 < 0$$

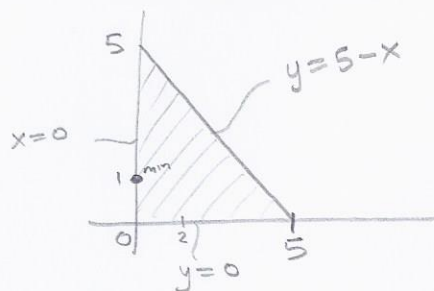
$\Rightarrow$   $(2, -1)$  is a saddle pt!

Test (-2, -1)

$$D = 0 - 4 < 0$$

$\Rightarrow$   $(-2, -1)$  is a saddle pt!

3. (10 points) Let  $f(x, y)$  be the function given in problem 2. Find the absolute maximum and minimum of  $f$  on the triangle with vertices  $(0,0)$ ,  $(0,5)$  and  $(5,0)$ .



The crit pts  $(2,-1)$  and  $(-2,-1)$  are not in the region. So test  $f(0,1) = -1$

Now test the boundary!

$$\underline{y=5-x}$$

$$f(x,y) \rightarrow f(x, 5-x) = g(x), \quad 0 \leq x \leq 5$$

$$\begin{aligned} g(x) &= (5-x)^2 + x^2(5-x) + x^2 - 2(5-x) \\ &= 25 - 10x + x^2 + 5x^2 - x^3 + x^2 - 10 + 2x \\ &= -x^3 + 7x^2 - 8x + 15, \quad 0 \leq x \leq 5 \end{aligned}$$

$$\begin{aligned} \Rightarrow g'(x) &= -3x^2 + 14x - 8, \quad 0 \leq x \leq 5 \\ &= -(3x-2)(x-4) \end{aligned}$$

$$\Rightarrow x = \frac{2}{3}, \quad x = 4 \text{ for crit. pts}$$

$$\begin{aligned} \text{So } g\left(\frac{2}{3}\right) &= f\left(\frac{2}{3}, \frac{13}{3}\right) = -\left(\frac{2}{3}\right)^3 + 7\left(\frac{2}{3}\right)^2 - 8\left(\frac{2}{3}\right) + 15 \\ &= -\frac{8}{27} + \frac{28}{9} - \frac{16}{3} + 15 \\ &= \frac{-8 + 84 - 144 + 405}{27} \\ &= \frac{337}{27} \approx 12.5 \end{aligned}$$

$$\begin{aligned} g(4) &= f(4, 1) = -4^3 + 7(4)^2 - 8(4) + 15 \\ &= -64 + 112 - 32 + 15 \\ &= 31 \end{aligned}$$

$$g(0) = f(0, 5) = 15$$

$$\begin{aligned} g(5) &= f(5, 0) = -5^3 + 7(25) - 8(5) + 15 \\ &= -125 + 175 - 40 + 15 \\ &= 25 \end{aligned}$$

$$\underline{y=0}$$

$$f(x,y) \rightarrow f(x, 0) = g(x), \quad 0 \leq x \leq 5$$

$$\begin{aligned} g(x) &= x^2 \\ \Rightarrow g'(x) &= 2x, \quad 0 \leq x \leq 5 \\ \Rightarrow x &= 0 \text{ for crit. pt.} \end{aligned}$$

$$\begin{aligned} \text{So } g(0) &= f(0, 0) = 0 \\ g(5) &= f(5, 0) = 25 \end{aligned}$$

$$\underline{x=0}$$

$$f(x,y) \rightarrow f(0, y) = g(y), \quad 0 \leq y \leq 5$$

$$\begin{aligned} g(y) &= y^2 - 2y, \quad 0 \leq y \leq 5 \\ \Rightarrow g'(y) &= 2y - 2 \\ \Rightarrow y &= 1 \text{ for crit. pt.} \end{aligned}$$

$$\begin{aligned} \text{So } g(1) &= f(0, 1) = -1 \\ g(0) &= f(0, 0) = 0 \\ g(5) &= f(0, 5) = 15 \end{aligned}$$

$\therefore f(4, 1) = 31 \rightarrow \text{abs max!}$   
 $f(0, 1) = -1 \rightarrow \text{abs min!}$



4. Let  $f(x, y, z) = x^3yz^2 - 4xy$ .

(a) (5 points) Find  $\nabla f = \langle 3x^2yz^2 - 4y, x^3z^2 - 4x, 2x^3yz \rangle$

(b) (10 points) Find the directional derivative of  $f$  at the point  $(1, -1, 2)$  in the direction  $\vec{v} = \langle 2, 0, -1 \rangle$

$$\nabla f(1, -1, 2) = \langle 3(1)^2(-1)(4) - 4(-1), 2^2 - 4(1), 2(1)^3(-1)(2) \rangle$$
$$= \langle -8, 0, -4 \rangle$$

$$\text{also } \vec{u} = \frac{\langle 2, 0, -1 \rangle}{\sqrt{4+1}} = \left\langle \frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}} \right\rangle$$

$$\begin{aligned} \Rightarrow D_{\vec{u}} f(1, -1, 2) &= \nabla f \cdot \vec{u} \\ &= \langle -8, 0, -4 \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}} \right\rangle \\ &= -\frac{16}{\sqrt{5}} + \frac{4}{\sqrt{5}} \\ &= \boxed{-\frac{12}{\sqrt{5}}} \end{aligned}$$

(c) (10 points) Find the equation of the tangent plane to the level surface  $f(x, y, z) = 0$  at the point  $(1, -1, 2)$ .

$$\vec{n} = \nabla f(1, -1, 2) = \langle -8, 0, -4 \rangle$$

$$\begin{aligned} \Rightarrow \text{tangent plane: } & \boxed{-8(x-1) - 4(z-2) = 0} \\ \text{or } & 2(x-1) + (z-2) = 0 \end{aligned}$$

(d) (10 points) Use differentials (that is, linear approximation) to approximate  $f(0.9, -0.9, 2.1)$

$$f(x, y, z) \approx L = -8(x-1) - 4(z-2) + f(1, -1, 2)$$

at the point  $(0.9, -0.9, 2.1)$

$$\begin{aligned} f &\approx -8(0.9-1) - 4(2.1-2) + (-4+4) \\ &= -8(-0.1) - 4(0.1) + 0 \\ &= 0.8 - 0.4 \end{aligned}$$

$$\boxed{= 0.4}$$

