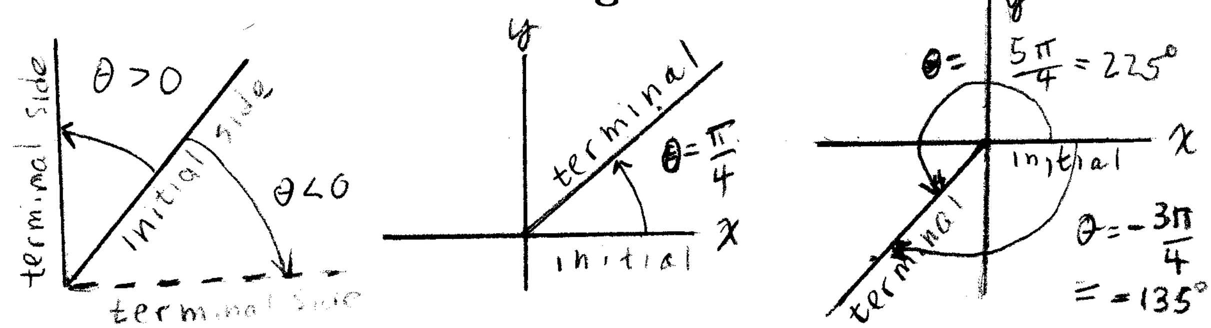
## Standard Position of an Angle




To consider an angle $\theta$ which is possibly negative, designate one side of the angle as the *initial* side. If $\theta$ is positive, the other side of the angle, called the *terminal* side, is obtained by rotating the initial side an amount $\theta$ counterclockwise about the vertex. If $\theta$ is negative, the rotation is an amount $|\theta|$ clockwise.

If we have a coordinate system, consisting of an $x$- and $y$-axis as introduced elementary algebra, we call this a *rectangular* coordinate system.

To draw an angle in *standard position* with respect to a rectangular coordinate system, move it so that the vertex is at the origin and the initial side of the angle is on the positive $x$-axis.

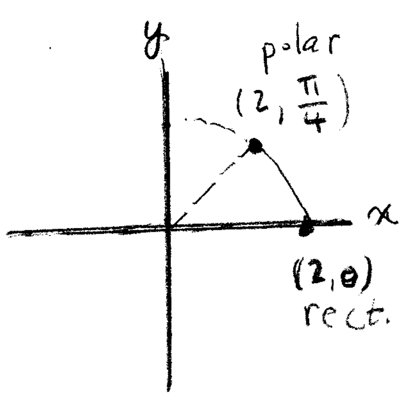
The middle diagram above shows an angle of $\pi/4$ drawn in standard position. The diagram at the right shows angles of $5\pi/4$ and $-3\pi/4$ drawn in standard position.

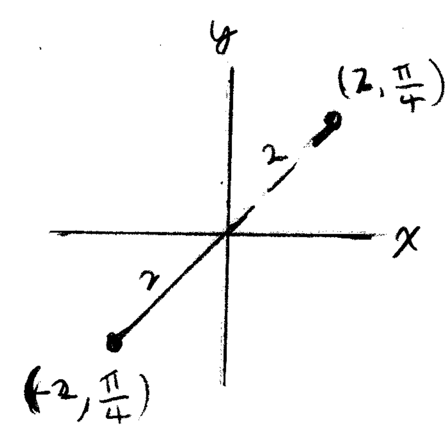
Angles $\theta_1$ and $\theta_2$ are called *coterminal* if, as in the diagram at the right, they have the same terminal side. Angles $\theta_1$ and $\theta_2$ are coterminal if and only if $\theta_2 = \theta_1 + 2n\pi$ for some integer $n$ (positive or negative).

## Polar Coordinates

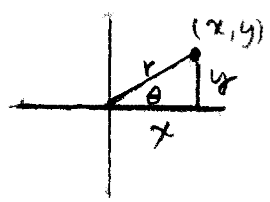
Given a point with rectangular coordinates, $(x, y)$, let $r$ be it's distance to the origin and let $\theta$ be an angle in standard position whose terminal side is the line segment from the origin to $(x, y)$. We call the pair $(r, \theta)$ *polar coordinates* of the point. Given any $r \geq 0$ and any real number $\theta$, the polar coordinates allow us to locate the point: Rotate the line segment from the origin to the point with rectangular coordinates $(r, 0)$ an angle $\theta$ about the origin. Then $(r, \theta)$ is the point to which rectangular $(r, 0)$ is rotated. Polar $(0, \theta)$ is the origin for all $\theta$. If $r$ is a negative real number, $(r, \theta)$ is defined to be the point with polar coordinates $(|r|, \theta + \pi)$, i.e. the reflection of $(|r|, \theta)$ across the origin.

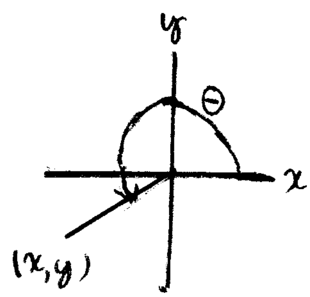
Although a point can only be described by only one pair $(x, y)$ of rectangular coordinates, any point can be described by infinitely many pairs $(r, \theta)$ of polar coordinates. We describe this situation by saying rectangular coordinates are in 1-1 correspondence with points in the plane, but polar coordinates are not.



We would like to establish the relation between the rectangular coordinates and the polar coordinates of any given point. We will first consider the relation when $\theta$ is an acute angle. In this case, we can use the trigonometry usually introduced in a ninth grade algebra course.




$(x, y)$ is 1-1
$(r, \theta)$ is not 1-1
for $r < 0$, $(r, \theta)$ means $(|r|, \theta+\pi)$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\frac{y}{x}$$

The equations in the second column are obtained by clearing parentheses in the first two equations in the first column, and the second equation in the third column follows from the third equation in the first column. The first equation in the third column follows from the Pythagorean theorem.

If $\theta$ is not acute, then, in trigonometry, $\cos\theta$, $\sin\theta$ and $\tan\theta$ are defined to be $\frac{x}{r}$, $\frac{y}{r}$ and $\frac{y}{x}$, respectively, so that equations above continue to hold.

The pairs of equations in the second and third columns are the conversions from polar coordinates to rectangular coordinates and vice versa.

By the graph of a polar equation we mean all points with polar coordinates $(r, \theta)$ which satisfy the equation. If, given an equation in rectangular coordinates, we find a polar equation with the same graph, we say we have converted the rectangular equation to a polar equation, or expressed the rectangular equation in terms polar coordinates. Analogously, we can express a polar equation in rectangular coordinates.

Example 1: Find the graph of $r = 2$. $r$ is the distance from the origin, so the graph is a circle of radius 2 centered at the origin.

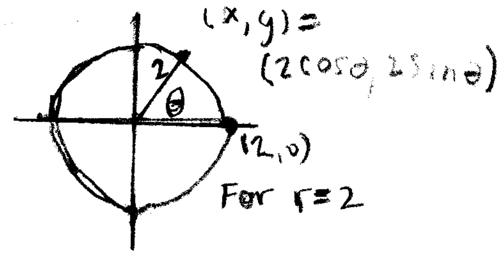
Example 2: Express the equation $x^2 + y^2 = 4$ in polar coordinates.

Solution: The most straightforward way to convert an equation to polar coordinates is to replace $x$ by $r\cos\theta$ and $y$ by $r\sin\theta$ wherever they occur. We substitute these and simplify:

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 4;\ r^2(\cos^2\theta + \sin^2\theta) = 4;\ r^2 = 4.$$

Either of $r = 2$ or $r = -2$ will give the complete circle of radius 2 and be considered a conversion.

The although direct substitution is the most straightforward method of conversion, when a rectangular equation contains the expression $x^2 + y^2$, it is easier to substitute $r^2$ and avoid the simplification we did above: $x^2 + y^2 = 4$; $r^2 = 4$.




We consider the graph in Examples 1 and 2 in more detail. We look at our polar conversion

$x = r\cos\theta$

$y = r\sin\theta$

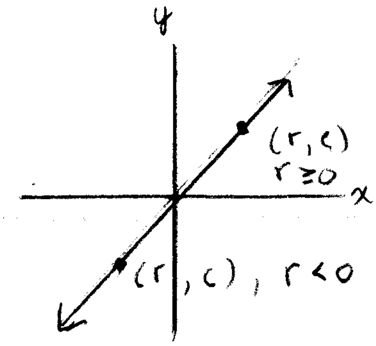
for $0 \leq \theta \leq 2\pi$. The graph at the right marks the point on the circle corresponding to a fixed value of $\theta$. If we draw the points for all $\theta = 0$ to $\theta = 2\pi$, we will draw the circle starting at the point $(2, 0)$ (rectangular coordinates) and progressing counterclockwise around the circle. If we think of $\theta$ being the time, measured from when we start, then we will be tracing the cirle at a uniform rate.

This is an example of what we call (from Section 11.1 in the text) *parametric equations* for the circle, and $\theta$ would be referred to as the *parameter*.

If, for $\theta$ ranging from 0 to $2\pi$, we graph $r = \pm 2$ for $\theta$ in increasing order, then $r = 2$ is graphed counterclockwise starting at $(x, y) = (2, 0)$, and $r = -2$ is graphed clockwise starting at $(-2, 0)$.

Example 3: Graph $(x^2 + y^2)^2 - 8(x^2 + y^2) + 16 = 0$.

Solution: One way to do this is to convert to polar coordinates: $(r^2)^2 - 8(r^2) + 16 = 0$. This is a perfect square: $(r^2 - 4)^2 = 0$, so $r^2 - 4 = 0$ and $r = 2$, our circle. The polar coordinates were not necessary; we could have written a perfect square in $x^2 + y^2$.




Example 4: Sketch the graph of $\theta = c$. Each point on the straight line through the origin making angle $\theta = c$ with the positive $x$-axis is the graph.

Example 5: Express the equation $y = 2x$ in polar coordinates.

Solution: Substitute:

$$r \sin\theta = 2r\cos\theta \qquad \text{or} \qquad r(\sin\theta - 2\cos\theta) = 0.$$

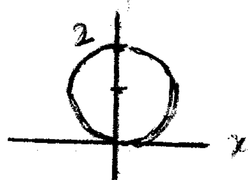
Thus either $r = 0$ which has graph just the origin $(0, 0)$ or $\sin\theta = 2\cos\theta$. Dividing both sides of the last equation by $\cos\theta$ yields $\tan\theta = 2$ and $\theta = \tan^{-1} 2$.

Example 6: Write $r = 2\sin\theta$ in rectangular coordinates. Sketch the graph.

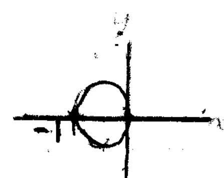
Solution: The result of the most straightforward substitution is not the easiest equation to deal with: $\sqrt{x^2+y^2} = 2\sin(\tan^{-1}\theta)$. However, if both sides of the polar equation are multiplied by $r$, the resulting equation is $r^2 = 2r\sin\theta$, which yields

$$x^2 + y^2 = 2y; \quad x^2 + y^2 - 2y = 0; \quad x^2 + (y-1)^2 = 1,$$

a circle of diameter 2 tangent to the $x$-axis at the origin with point farthest from the origin on the positive $y$-axis.




Example 6




Example 7

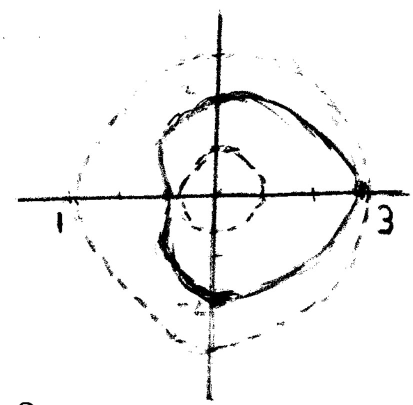
Example 7: Graph $r = -\cos\theta$.

Solution: The same procedure here leads to the rectagular form $(x+\frac{1}{2})^2 + y^2 = \frac{1}{4}$, and the graph is a circle of diameter 1 tangent to the $x$-axis at the origin with point farthest from the origin on the negative $x$-axis.

It is useful to remember that the graph of $r = a\cos\theta$ $[r = a\sin\theta]$ is a circle of diameter $|a|$ tangent at the origin to the $x$-axis [$y$-axis] and having point farthest from the origin on the positive axis if $a > 0$ and negative axis if $a < 0$.

A graph of the form $r = a + b\sin\theta$ or $r = a + b\cos\theta$ is called a *limaçon*. The following three examples illustrate what the graph may look like. This also illustrates the most elementary technique for graphing a polar equation and the fact that it may be easier to graph an equation in polar form than in rectangular form.

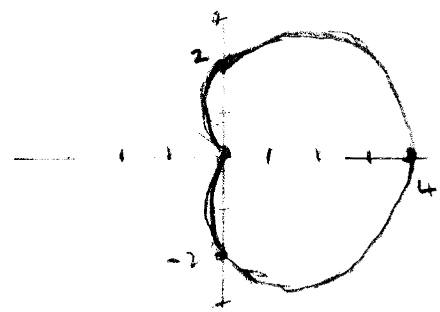
Example 8: Case 1: $|a| > |b|$
Graph $r = 2 + \cos\theta$.




Solution: The largest value of $r$ is 3, which occurs when $\cos\theta = 1$ at $\theta = 0$, and the smallest value of $r = 1$, when $\theta = \pi$. This tells us the entire graph between the circles $r = 1$ and $r = 3$. These circles are shown with dotted lines. They are not part of the graph. As $\theta$ increases from 0 to $\pi/2$, the value of $r$ decreases from 3 to 2. As $\theta$ goes from $\pi/2$ to $\pi$, $r$ continues to decrease to its smallest value, 1.

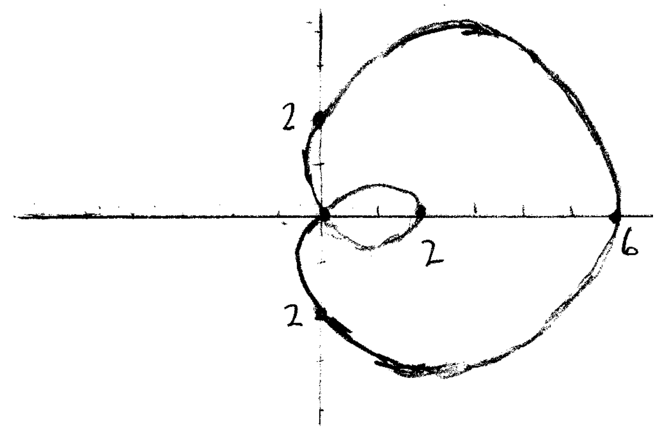
We could continue drawing arcs of the graph for $\pi \leq \theta \leq 3\pi/2$ and $3\pi/2 \leq \theta \leq 2\pi$ to complete the graph. However, it is easier to make use of symmetry. Since $\cos(-\theta) = \cos\theta$, $r(-\theta) = r(\theta)$. The angle $-\theta$ is reflection across the $x$-axis of the angle $\theta$. A pair of the symmetrically placed points is shown on the graph.

Compare this to graphing the rectangular form. If we multiply both sides by $r$, transpose the resulting second term on the right to the left side, and square both sides, we obtain $(r^2 - r\cos\theta)^2 = (2r)^2$, which is $[(x^2 + y^2 - x]^2 = 4(x^2 + y^2)$ in rectangular form.




Example 9: Case 2: $|a| = |b|$
Graph $r = 2 + 2\cos\theta$.

Solution: The analysis is exactly the same as in Example 8 except that the minimum value of $r$, again at $\theta = \pi$, is 0.

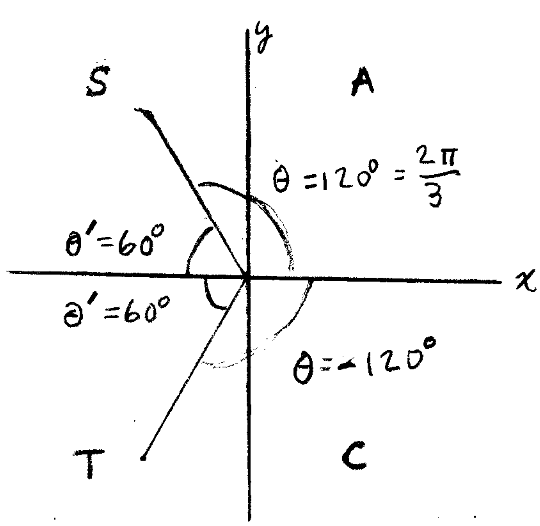



Example 10: Case 3: $|a| < |b|$
Graph $r = 2 + 4\cos\theta$.

Solution: Now the value of $\theta$ for which $r$ first reaches zero, signifying the graph is at the origin, is less than $\pi$:

$$0 = r = 2 + 4\cos\theta; \qquad \cos\theta = -\frac{1}{2}.$$

The reference angle is $\pi/3$ and cos is negative in the second and third quadrants, so the smallest positive value of $\theta$ is $2\pi/3$. From $2\pi/3$ to $\pi$ $r$ continues to decrease, reaching its minimum value $r(\pi) = 2 - 4 = -2$. And again, the second half of the graph is the mirror image across the the $x$-axis of the first half




$\cos\theta = -\frac{1}{2} = -\cos\theta'$

$\theta' = 60° = \frac{\pi}{3}$

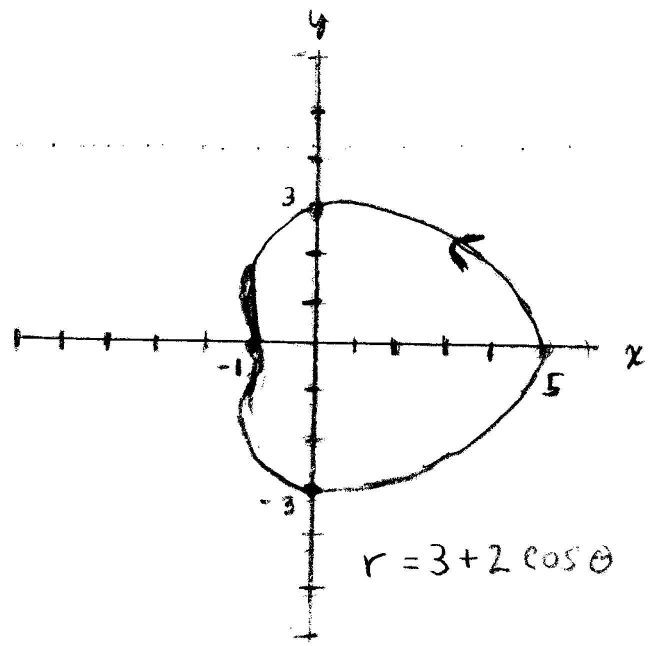
You should remember the general shape of the graphs for each of the three cases. Then, to graph a limaçon, just graph the $x$- and $y$-intercepts, namely $r(0)$, $r(\frac{\pi}{2})$, $r(\pi)$, and $r(\frac{3\pi}{2})$, and, in Case [(2) and] (3), the origin. Then sketch the graph through successive points, starting at the point farthest from the origin. Below are several examples.

Example 11: Graph
$r = 3 + 2\cos\theta$.

$r(\frac{\pi}{2}) = r(\frac{3\pi}{2}) = 3$
$r(0) = 5$; $r(\pi) = 1$

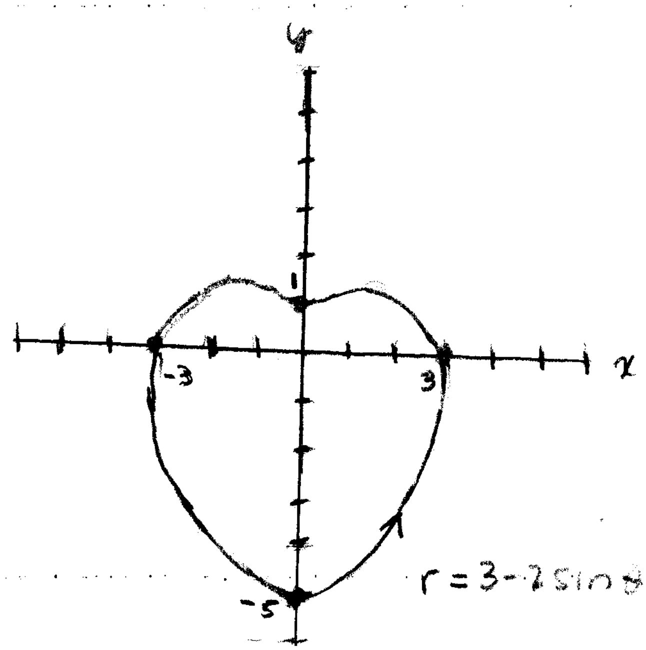
Example 12: Graph
$r = -3 + 2\cos\theta$.

$r(\frac{\pi}{2}) = r(\frac{3\pi}{2}) = -3$
$r(0) = -1$; $r(\pi) = -5$
Same points as above, with a different description, so the same graph.



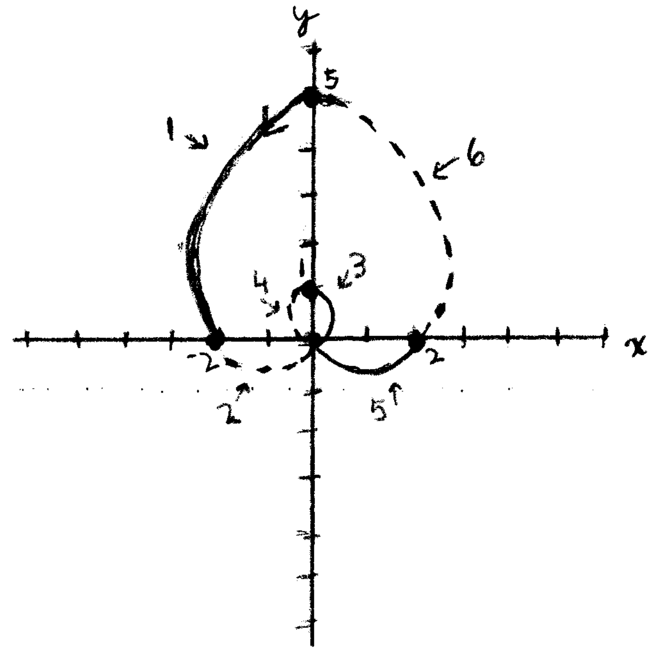

Example 13: Graph
$r = 3 - 2\sin\theta$.

$r(0) = r(\pi) = 3$
$r(\frac{\pi}{2}) = 1$; $r(\frac{3\pi}{2}) = 5$



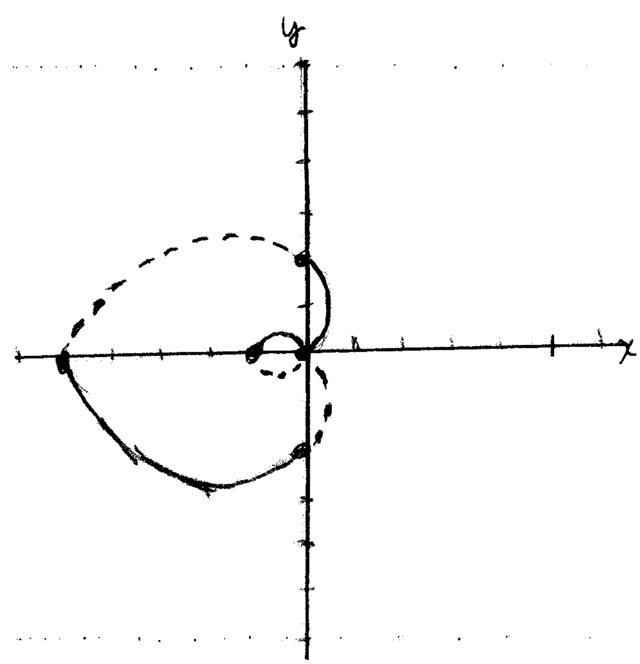
Example 14: Graph
$r = 2 + 3\sin\theta$.

$r(0) = r(\pi) = 2$
$r(\frac{\pi}{2}) = 5$; $r(\frac{3\pi}{2}) = -1$




Example 15: Graph
$r = 2 - 3\cos\theta$.

$r(\frac{\pi}{2}) = r(\frac{3\pi}{2}) = 2$
$r(0) = -1$; $r(\pi) = 5$



## Optional Topic

A graph of an equation of the form $r = a\sin n\theta$ or $r = a\cos n\theta$, where $a$ is a non-zero real number and $n$ is an integer greater than or equal to 2 is called a *rose leaf curve* (some variants of the name and defnition are used). We will be concerned only with $n = 2$ and $n = 3$.

Example 16: Graph $r = 3\sin 2\theta$. $|r|$ will take on its maximum value when $\sin 2\theta = \pm 1$ or

$$2\theta = \tfrac{\pi}{2} + 2n\pi \quad \text{or} \quad 2\theta = \tfrac{3\pi}{2} + 2n\pi.$$

The first four positive values of $\theta$ are $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

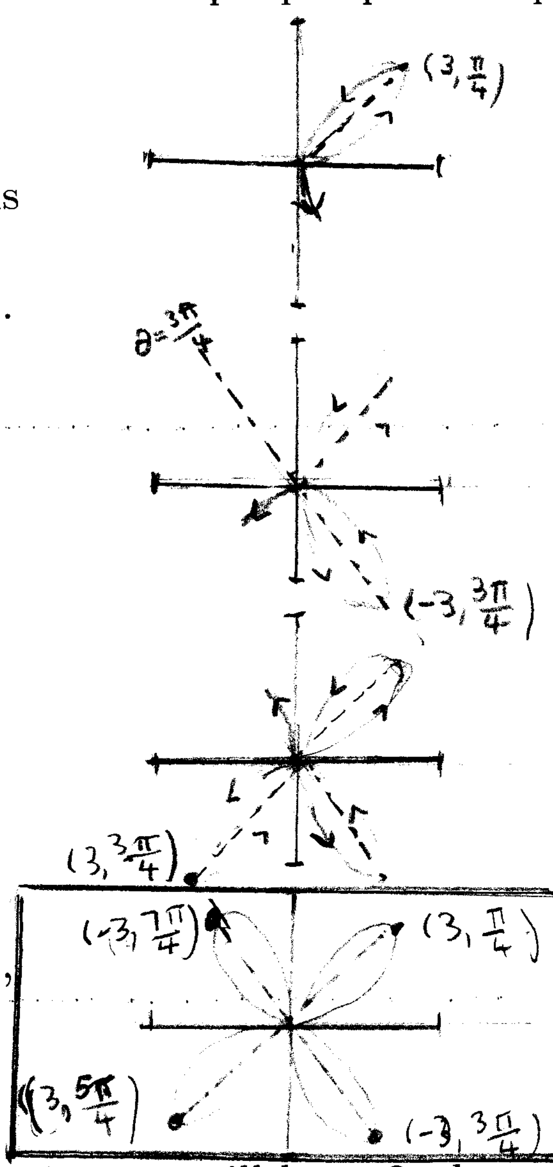
As $\theta$ increases from 0 to $\frac{\pi}{4}$, $r$ increases from 0 to 3 forming half of a leaf with tip at $\frac{\pi}{4}$.

The second half of the leaf is obtained as $\theta$ continues to increase to $\frac{\pi}{2}$, where $r = 0$ again.

For values of $\theta$ increasing from $\frac{\pi}{2}$ to $\pi$, $\sin 2\theta$ is negative, and the second leaf is traced, with tip at $\frac{3\pi}{4}$.

For values of $\theta$ increasing from $\pi$ to $\frac{3\pi}{2}$, $\sin 2\theta$ is positive, and the third leaf is traced, with tip at $\frac{5\pi}{4}$.

For values of $\theta$ increasing from $\frac{3\pi}{2}$ to $2\pi$, $\sin 2\theta$ is negative, and the second leaf is traced, with tip at $\frac{7\pi}{4}$.



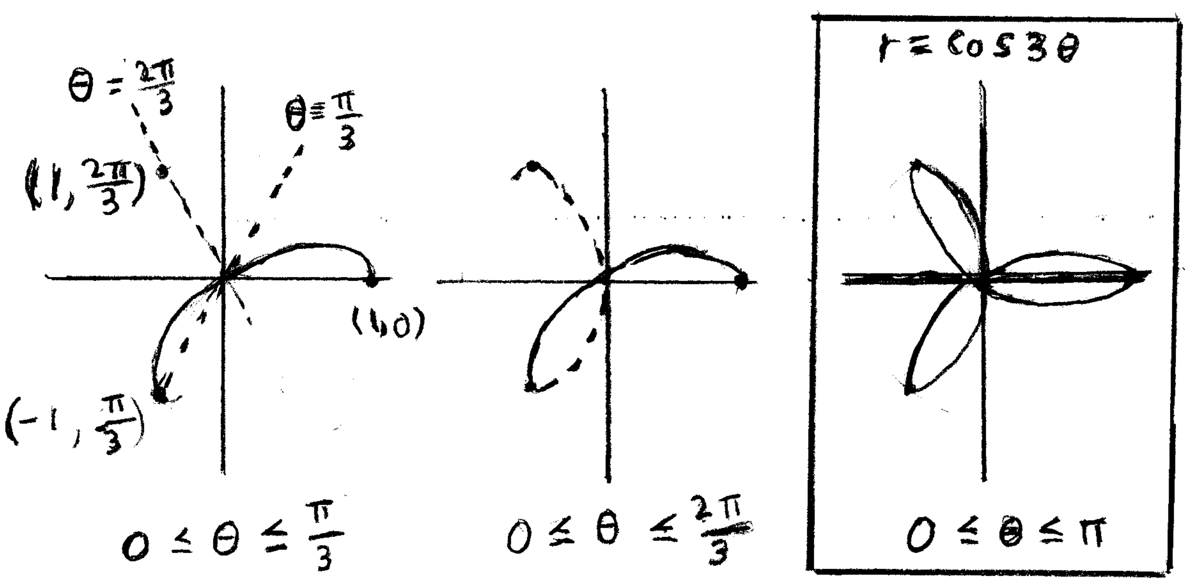
Example 17: Graph $r = \cos 3\theta$.

Tips of leaves correspond to values of $\theta$ for which $|r| = 1$, that is for $\cos 3\theta = \pm 1$, so

$$3\theta = 2n\pi \quad \text{or} \quad 3\theta = \pi + 2n\pi.$$

The first three non-negative values of $\theta$ are 0, $\frac{\pi}{3}$, and $\frac{2\pi}{3}$. The three tips of leaves are at points with polar coordinates $(1, 0)$, $(-1, \frac{\pi}{3})$, and $(1, \frac{2\pi}{3})$. After three leaves, corresponding to graphing for $0 \leq \theta \leq \pi$, the graph will repeat. The graph below on the right shows the complete rose leaf curve.

The leaf with tip at $(1, 0)$ could be described as the portion of the graph corresponding to $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$.




A rose leaf curve for which $n$ is odd will have $n$ leaves.

## Section 11.5

Section 11.5 is part of the syllabus only as far as Example 2. Two additonal examples are given below.

Example 1: Graph the limaçon $r = 3 - 2\sin\theta$, and find the area of the portion of the region inside the graph which is in the first quadrant.

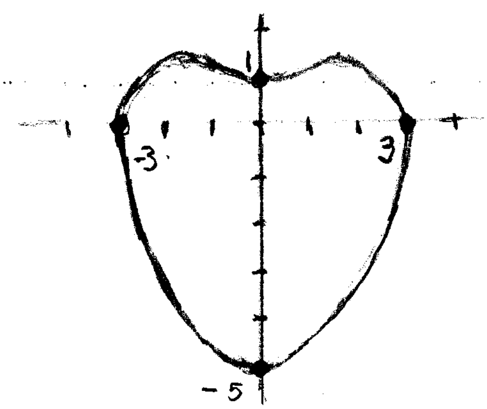
We reproduce the sketch of the graph shown in Example 13.

$r(0) = 3$

$r(\frac{\pi}{2}) = 2$

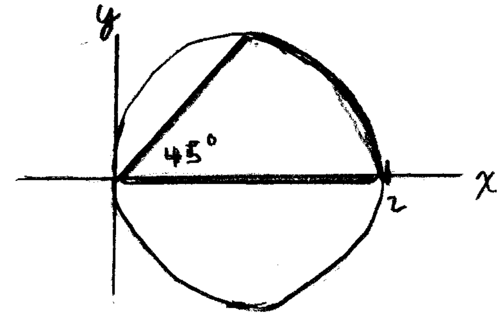
$r(\pi) = 3$

$r(\frac{3\pi}{2}) = 6$




$$
\begin{aligned}
A =& \frac{1}{2}\int_a^b r^2\, d\theta \\
=& \frac{1}{2}\int_0^{\pi/2} (3 - 2\sin\theta)^2\, d\theta \\
=& \frac{1}{2}\int_0^{\pi/2} 9 - 12\sin\theta + 4\sin^2\theta\, d\theta \\
=& \frac{9\pi}{4} + 6\cos\theta|_0^{\pi/2} + \int_0^{\pi/2} 1 - \cos 2\theta\, d\theta \\
=& \frac{9\pi}{4} - 6 + \frac{\pi}{2} - \frac{1}{2}\sin 2\theta|_0^{\pi/2} \\
=& \frac{11\pi}{4} - 6.
\end{aligned}
$$

Example 2: If $AB$ is the diameter of a circle of radius 1 and $C$ is a point on the circle such that the inscribed $\angle CAB = 45°$, what is the area bounded by the sides of the angle and the (smaller) intercepted arc?




Set up a coordinate sytem with $A$ as the origin and $AB$ on the $x$-axis. In this coordinate system, the circle is the graph of $r = 2\cos\theta$, so the area is

$$A = \frac{1}{2}\int_0^{\pi/4} 4\cos^2\theta\, d\theta = \int_0^{\pi/4} 1 + \cos 2\theta\, d\theta$$

$$= \frac{\pi}{4} + \frac{1}{2}\sin 2\theta\Big|_0^{\pi/4} = \frac{\pi}{4} + \frac{1}{2}.$$

## Review

**1.** Change from polar coordinates to rectangular coordinates and vice versa:

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\theta \qquad \theta = \tan^{-1}\frac{y}{x}$$

**2.** Know the shapes of and be able to graph:

(a) $r = a$: a circle of radius $a$ centered at the origin

(b) $r = \theta$: a straight line through the origin making angle $\theta$ with the positive $x$-axis

(c) $r = a\cos\theta$ and $r = a\sin\theta$: a circle tangent to a coordinate axis at the origin

(d) $r = a + b\sin\theta$ and $r = a + b\cos\theta$: a limacon

**3.** The area of a region bounded by $\theta = a$, $\theta = b$, and $r = f(\theta)$ is

$$A = \frac{1}{2}\int^{b} f(\theta)^2\, d\theta.$$