MATHEMATICS 39104 FINAL EXAM Fall, 2014

Part I Answer all questions in this part.

(1) Compute the general solution of each of the following (8 points each):

(a) $x\,\dfrac{dy}{dx} + xy \quad = 1 - y$. *Linear*: $\frac{dy}{dx} + (1 + \frac{1}{x})y = \frac{1}{x}$. Integrating factor

$exp[x + \ln(x)] = xe^x$. $[xe^x y]' = e^x$. So $y = [e^x + C]/xe^x$.

(b) $(x - y)\ \dfrac{dy}{dx} \quad = \quad (y\ +\ x)$.

Homogeneous (not Exact): With $z = \frac{y}{x}$, $xz = y$. So $x\frac{dz}{dx} + z = \frac{z+1}{1-z}$. $x\frac{dz}{dx} = \frac{z^2+1}{1-z}$.
$\int \frac{1-z}{1+z^2}\ dz = \int \frac{dx}{x}$. $\arctan(\frac{y}{x}) - \frac{1}{2}\ln(1 + \frac{y^2}{x^2}) = \ln(x) + C$.

(2) Solve the following initial value problems (8 points each):

(a) First, find the value of b such that the following differential equation is exact. Then solve the initial value problem with that value of b. (Leave your answer in implicit form.)

$$(xy^2 + bx^2y)\ dx\ +\ (x^3 + x^2y)\ dy\ =\ 0, \quad y(2)\ =\ 3.$$

$\frac{\partial}{\partial y}(xy^2 + bx^2y) = 2xy + bx^2$

$\frac{\partial}{\partial x}(x^3 + x^2y) = 2xy + 3x^2$
So they are equal, and so the system is *Exact* when $b = 3$
$F = \int\ (xy^2 + 3x^2y)\ dx = x^2y^2/2 + x^3y + h(y)$
So $\frac{\partial F}{\partial y} = x^2y + x^3 + h'(y) = x^3 + x^2y$.
$h'(y) = 0$ and so up to a constant $h(y) = 0$.
General solution $x^2y^2/2 + x^3y = C$. When $x = 2, y = 3$ so $C = 42$.

(b) $y''\ -\ 3y'\ +\ 3y\ \ =\ \ 0 \qquad y(0) = 1,\ y'(0) = 0.$

If $r^2 - 3r + 3 = 0$ then $r = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$
$y = C_1e^{3x/2}\cos(\sqrt{3}x/2)\ +\ C_2e^{3x/2}\sin(\sqrt{3}x/2)$.
$y' = C_1[\frac{3}{2}e^{3x/2}\cos(\sqrt{3}x/2) - \frac{\sqrt{3}}{2}e^{3x/2}\sin(\sqrt{3}x/2)]$
$+C_2[\frac{3}{2}e^{3x/2}\sin(\sqrt{3}x/2) + \frac{\sqrt{3}}{2}e^{3x/2}\cos(\sqrt{3}x/2)]$.

$y(0) = 1$ implies $C_1 = 1$ and $y'(0) = 0$ then implies $C_2 = -\sqrt{3}$.

(3) Compute the general solution of each of the following (8 points each):

$$\text{(a)} \quad x^2 \frac{dy}{dx} - 3y = 0.$$

Variables Separable: $\ln(y) = -\frac{3}{x} + C$, $y = Ce^{-3/x}$.

$$\text{(b)} \quad y'' + 4y = \tan(2t).$$

If $r^2 + 4 = 0$, $r = \pm 2$, $y_h = C_1 \cos(2t) + C_2 \sin(2t)$. *Variation of Parameters:* $y_p = u_1 \cos(2t) + u_2 \sin(2t)$
$u_1' \cos(2t) + u_2' \sin(2t) = 0$,
$-u_1' \sin(2t) + u_2' \cos(2t) = \tan(2t)$.
Wronskian $= 2$.
$u_1 = -\frac{1}{2} \int [\tan(2t) \sin(2t)] dt = -\frac{1}{2} \int [(1 - \cos^2(2t)) / \cos(2t)] dt$
$= -\frac{1}{2} \int [\sec(2t) - \cos(2t)] dt = \frac{1}{4} [\ln |sec(2t) + \tan(2t)| - \sin(2t)]$.
$u_2 = \frac{1}{2} \int [\tan(2t) \cos(2t)] dt = \frac{1}{2} \int [\sin(2t)] dt = -\frac{1}{4} \cos(2t)$.
$y_p = \frac{1}{4} [\ln |sec(2t) + \tan(2t)| \cdot \cos(2t)$.

(4) For each of the 2×2 systems $\frac{dX}{dt} = AX$ with the matrix A given below, compute the eigenvalues of A and fundamental matrix solution e^{tA}. For each determine whether the equilibrium at $X = 0$ is stable, unstable or a saddle.

$$A = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} \qquad A = \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}.$$

(10 points)

(You should be able to write down the solutions without computing the eigenvectors.)

Type 2:If $A = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}, e^{tA} = \begin{pmatrix} e^{3t} & 0 \\ te^{3t} & e^{3t} \end{pmatrix}$. The characteristic equation is $(r - 3)^2 = 0$, $r = 3, 3$ unstable equilibrium.

Type 3:If $A = \begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix} .e^{tA} = e^{-3t} \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}$. The characteristic equation is $(r + 3)^2 + 4 = 0$, $r = -3 \pm 2i$ stable equilibrium.

(5) Assume that y_1 and y_2 are solutions of the equation $y'' + p(x)y' + q(x)y = 0$ where $p(x)$ and $q(x)$ are continuous for all x in the real line. Show that $y_h = C_1 y_1 + C_2 y_2$ is a solution for any choice of constants C_1 and C_2. Assume that the Wronskian at $x = 0$ is nonzero.

$(C_1y_1 + C_2y_2)'' + p(x)(C_1y_1 + C_2y_2)' + q(x)(C_1y_1 + C_2y_2) =$
$C_1(y_1'' + p(x)y_1' + q(x)y_1) + C_2(y_2'' + p(x)y_2' + q(x)y_2) = 0.$

(i) Explain why you can choose constants C_1 and C_2 so that y_h is a solution with the initial conditions $y(0) = A$ and $y'(0) = B$ for any choice of real numbers A and B. (12 points)

Because the Wronskian is nonzero, the system
$C_1y_1(0) + C_2y_2(0) = A$
$C_1y_1'(0) + C_2y_2'(0) = B$
has a solution for any A, B by Cramer's Rule.

(ii) Explain how you know that every solution is of the form $y_h = C_1y_1 + C_2y_2$.

Among these solutions is a solution for every initial value problem and every IVP has a unique solution. Any solution solves some IVP and so is one of these.

PartII Answer all sections of three (3) questions out of the five (5) questions in this part (10 points each).

(6) (a) For the equation $2t^2y'' - 3ty' + 3y = 0 \quad (t > 0)$, $y_1(t) = t$ is a solution.
(i) Using any method, obtain a second, independent solution.

Euler Equation: $2r(r-1) - 3r + 3 = (2r-3)(r-1) = 0$. $r = 1, \frac{3}{2}$. So $t, t^{3/2}$ are independent solutions.
Alternatively, use reduction of order with $y_2 = ut$.
$y_2 = ut, y_2' = u + u't, y_2'' = 2u' + u''t$. So $0 = 2t^2y_2'' - 3ty_2' + 3y_2 = u't^2 + 2u''t^3$.
Let $v = u', v' = u''$. $\ln(v) = -\frac{1}{2}\ln(t), u' = t^{-1/2}, u = 2t^{1/2}$ So $y_2 = 2t^{3/2}$

(ii) Compute the Wronskian of the pair of solutions.
$t \cdot \frac{3}{2}t^{1/2} - 1 \cdot t^{3/2} = \frac{1}{2}t^{3/2}$.

(b) Suppose that a mass weighing 10 pounds stretches a vertical spring 6 inches. Set up an initial value problem for y(t), the displacement of the mass from its equilibrium position after t seconds, if the mass is set in motion from its equilibrium position with an initial velocity of 4 feet per second downward. (Assume the acceleration due to gravity is 32 feet per second per second and assume there is no friction). You need not solve the equation.

$m = 10/32, k \cdot \frac{1}{2} = 10.$
$\frac{10}{32}u'' + 20u = 0,\ u(0) = 0, u'(0) = -4.$

(7) (a) Compute the general solution of the differential equation

$$y^{(6)} - y^{(3)} \quad = \quad 0.$$

$r^6 - r^3 = r^3(r^3 - 1)$. roots are $0, 0, 0$ and by De Moivre's Theorem $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Alternatively, $r^3 - 1 = (r-1)(r^2 + r + 1)$ and use the Quadratic Formula.
$y = C_1 + C_2t + C_3t^2 + C_4e^t + C_5e^{-t/2}\cos(\sqrt{3}t/2) + C_6e^{-t/2}\sin(\sqrt{3}t/2)$

(b) Determine the test function $Y(t)$ with the fewest terms to be used to obtain a particular solution of the following equation via the *method of undetermined coefficients*. Do not attempt to determine the coefficients.

$$y^{(6)} - y^{(3)} \quad = \quad 7 + t\sin(\frac{\sqrt{3}}{2}\ t) + t^2e^{-t/2} + e^{-t/2}\cos(\frac{\sqrt{3}}{2}\ t) - \cos(\frac{\sqrt{3}}{2}\ t)$$

Associated roots are:$0, \pm i\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \pm i\frac{\sqrt{3}}{2}$.
$y_p = t^3[A] + [(Bt + C)\cos(\frac{\sqrt{3}}{2}\ t) + (Dt + E)\sin(\frac{\sqrt{3}}{2}\ t)]$
$+\ t[Fe^{-t/2}\cos(\frac{\sqrt{3}}{2}\ t) + Ge^{-t/2}\sin(\frac{\sqrt{3}}{2}\ t)$.

(8) For the differential equation:

$$3y'' \ + \ (x^3 - 3x)y' \ + \ (1 + 3x^2)y \quad = \quad 0$$

(a) Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first three nonzero terms of the solution with $y(0) = 0, y'(0) = 27$.

$a_{k+2} = \frac{1}{3(k+2)(k+1)}[-(k-2)a_{k-2} + 3ka_k - a_k - 3a_{k-2}]$
$= \frac{1}{3(k+2)(k+1)}[(3k-1)a_k - (k+1)a_{k-2}]$
$a_0 = 0, a_1 = 27, (k = 0)a_2 = 0, (k = 1)a_3 = 3, (k = 2)a_4 = 0, (k = 3)a_5 = -7/5$. $y = 27x + 3x^3 - 7x^5/5 + ...$

(b) Show that the solution given in (a) is an odd function (Hint: what is a_n when n is even ?)

$a_n = 0$ for all even n.

(9) The operator L on infinitely differentiable functions y (with domain $x > 0$) is defined by $L(y)(x) \;=\; x^2y'(x)$.

(a) Show that L is a linear operator.

$L(Cy_1 + y_2) = x^2(Cy_1 + y_2)' = Cx^2y_1' + Cx^2y_2' = CL(y_1) + L(y_2)$.

(b) Compute an eigenvector with eigenvalue 3 for this operator.

$x^2y' = 3y$ is Problem 3(a). $y = e^{-1/x}$ is an eigenvector with eigenvalue 3.

(10) (a) Compute the Fourier series for the function f such that $f(x) = x^2$ on the interval $[-1, 1]$ and $f(x + 2) = f(x)$ for all real x.

The function is even and so all $b_n = 0$. $a_0 = 2\int_0^1 x^2dx = 2/3$.
$a_n = 2\int_0^1 x^2\cos(n\pi x)dx = 2([x^2][\frac{1}{n\pi}\sin(n\pi x)] - [2x][-(\frac{1}{n\pi})^2\cos(n\pi x) + [2][-(\frac{1}{n\pi})^3\sin(n\pi x)]\ |_0^1) = \frac{4(-1)^n}{(n\pi)^2}$.
$f(x) = \frac{1}{3} + \Sigma_1^\infty \frac{4(-1)^n}{(n\pi)^2}\cos(n\pi x)$.

(b) Compute the solution to the partial differential equation with x in the interval $[0, \pi]$ and $t > 0$:

$$\begin{aligned} u_t &= 16u_{xx} \quad \text{with} \\ u(t,0) = u(t,\pi) &= 0 \quad \text{for } t > 0 \qquad \text{(boundary conditions)} \\ u(0,x) &= 5\sin(x) - \sin(5x) \quad \text{for } 0 < x < \pi \qquad \text{(initial conditions)} \end{aligned}$$

$u(t,x) = 5e^{-(4)^2t}\sin(x) - e^{-(4\cdot 5)^2t}\sin(5x)$.