Math 39104 - Notes on Differential Equations and Linear Algebra

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These notes are a supplement for Math 39104 to the book *Elementary Differential Equations and Boundary Value Problems* by Boyce and DiPrima. At the end of the notes you will find the detailed syllabus and homework assignments.

1 First Order Differential Equations

A first order differential equation is an equation of the form

$$x' = \frac{dx}{dt} = f(x,t). \tag{1.1}$$

It describes that rate at which a point at x is moving on the real line $\mathbb{R}$ at time t. A *solution* is a function $x(t)$ which satisfies the equation.

While we often interpret a differential equation this way, in terms of motion, what is needed is to keep track of which is the independent variable (in this case t) and which is the dependent variable (in this case x). Almost as often as this, we will see the equation written as $y' = \frac{dy}{dx} = f(y,x)$. with independent variable x and dependent variable y, but any letter can be used for either variable.

The simplest case is when $f(y,x)$ depends only on x. That is, we are given the derivative and solve by integrating. When we integrate we get $y = \int f(x)dx + C$ where C is the arbitrary constant of integration. This illustrates that a differential equation usually has infinitely many solutions. In this case, changing C moves the graph up or down and a particular solution is determined by specifying a point (x_0, y_0) through which it passes. In general, an *initial value problem* (= IVP) is a differential equation together with a choice of such a point.

$$\frac{dy}{dx} = f(y,x). \qquad \text{and} \qquad y(x_0) = y_0. \tag{1.2}$$

The name "initial value problem" comes from the fact that often the initial value for (1.1) is given by specifying $x = x_0$ at the time $t = 0$.

For example, *linear growth* is given by $\frac{dx}{dt} = m$ with the rate of change constant. If $x(0) = x_0$ then the solution is $x = x_0 + mt$.

The differential equation is called *autonomous* when $f(x,t)$ depends only on x. That is, the rate of change depends only upon the current position and not the time. Thus, $\frac{dx}{dt} = f(x)$. For such an equation the roots of f play a special role. If $f(e) = 0$ then $x = e$ is called an *equilibrium* for the equation because the constant function $x(t) = e$ is then a solution. The most important, and simplest, example is *exponential growth* given by $\frac{dx}{dt} = rx$ and so when $x \neq 0, \ln|x|$ changes linearly. That is, if $x(0) = x_0$ then $\ln|x| = \ln|x_0| + rt$. Exponentiating we get $x = x_0 e^{rt}$. Notice that this description includes the equilibrium solution $x = 0$ with $x_0 = 0$. When $r < 0$ the equilibrium at 0 is an *attractor*. Solutions starting near 0 (and in this case all solutions) move toward this equilibrium with limit 0 as $t \to \infty$. When $r > 0$ the solution is a *repellor*. Solutions starting near 0 move away from it.

When f has more than one root then the system has more than one equilibrium. On each of the intervals between two equilibria, f is either strictly positive or strictly negative and so a point in such an interval moves up or down towards the equilibria at the end-point. Consider the examples:

$$\begin{aligned} &\text{(i)} \qquad \frac{dx}{dt} = -x + x^3. \\ &\text{(ii)} \qquad \frac{dx}{dt} = \begin{cases} x^2\cos(1/x) & x \neq 0, \\ 0 & x = 0. \end{cases} \end{aligned} \tag{1.3}$$

Exercises 1.1. For each of the following autonomous equations, draw a diagram marking the equilibria and the direction of motion in the complementary intervals. Indicate which equilibria are attracting and which are repelling.

$$(a) \quad \frac{dx}{dt} = 2x - x^2.$$

$$(b) \quad \frac{dx}{dt} = (x^4 - x^3 - 2x^2)(x - 5).$$

2 Vectors and Linear Maps

You should recall the definition of a vector as an object with magnitude and direction (as opposed to a scalar with magnitude alone). This was devised as a means of representing forces and velocities which exhibit these vector characteristics. The behavior of these physical phenomena leads to a geometric notion of addition of vectors (the "parallelogram law") as well as multiplication by scalars - real numbers. By using coordinates one discovers important properties of these operations.

Addition Properties

$$\begin{aligned} (\mathbf{v} + \mathbf{w}) + \mathbf{z} &= \mathbf{v} + (\mathbf{w} + \mathbf{z}). \\ \mathbf{v} + \mathbf{w} &= \mathbf{w} + \mathbf{v}. \\ \mathbf{v} + \mathbf{0} &= \mathbf{v}. \\ \mathbf{v} + -\mathbf{v} &= \mathbf{0}. \end{aligned} \tag{2.1}$$

The first two of these, the Associative and Commutative Laws, allow us to rearrange the order of addition and to be sloppy about parentheses just as we are with numbers. We use them without thinking about them. The third says that the zero vector behaves the way the zero number does: adding it leaves the vector unchanged. The last says that every vector has an "additive inverse" which *cancels* it.

Scalar Multiplication Properties

$$\begin{aligned} a(b\mathbf{v}) &= (ab)\mathbf{v}. \\ 1\mathbf{v} &= \mathbf{v}. \end{aligned} \tag{2.2}$$

The first of these looks like the Associative Law but it isn't. It relates multiplication between numbers and scalar multiplication. The second says that multiplication by 1 leaves the vector unchanged the way the addition with $\mathbf{0}$ does.

Distributive Properties

$$\begin{aligned} a(\mathbf{v} + \mathbf{w}) &= a\mathbf{v} + a\mathbf{w}. \\ (a + b)\mathbf{v} &= a\mathbf{v} + b\mathbf{v}. \end{aligned} \tag{2.3}$$

These link addition and scalar multiplication.

In Linear Algebra we abstract from the original physical examples. Now a vector is just something in a vector space. A vector space is a collection of objects which are called vectors. What is special about a vector space is that it carries a definition of addition between vectors and a definition of multiplication of a vector by a scalar. For us a scalar is a real number, or possibly a complex number. We discard the original geometric picture and keep just the properties given above. These become the *axioms of a vector space*. Just from them it is possible to derive various other simple, but important properties.

$$\begin{aligned} a\mathbf{v} = \mathbf{0} \quad &\Longleftrightarrow \quad a = 0 \quad \text{or} \quad \mathbf{v} = \mathbf{0}. \\ (-1)\mathbf{v} &= -\mathbf{v}. \end{aligned} \tag{2.4}$$

We won't bother with the derivation because such properties will be obvious in the examples we will consider.

Those examples will either be lists of numbers or real-valued functions. For example, the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ were described in Math 203. These consists of all lists of two or three numbers, but such a list could have any size (or shape). You should be able to imagine for yourself how you add vectors and multiply by scalars in $\mathbb{R}^{17}$. For real-valued (or complex valued) functions we define

$$\begin{aligned} (f + g)(x) &= f(x) + g(x). \\ (af)(x) &= a(f(x)). \end{aligned} \tag{2.5}$$

These again look like the distributive law and the associative law, but they are not. Instead, starting with two functions f and g the right hand side in each case is used to define the new functions labeled $f + g$ and af.

Given a finite list of vectors, like $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, it is possible to build from them infinitely many vectors from them by using the vector space operations. A *linear combination* of them is a vector $C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + C_3\mathbf{v}_3$ obtained by choosing scalars C_1, C_2, C_3.

A set of vectors in a vector space is a *subspace* when it is closed under addition and scalar multiplication. In that case, it is a vector space in its own right. For example the functions which can be differentiated infinitely often form a subspace and the polynomials are a subspace of it.

Exercises 2.1. Which of the following are subspaces? Justify your answers.

(a) The set of real-valued functions f such that $f(17) = 0$.

(b) The set of real-valued functions f such that $f(0) = 17$.

(c) The set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

A *linear operator* or *linear transformation* is just a function L between vector spaces, that is, the inputs and outputs are vectors, which satisfies *linearity*, which is also called the *superposition property*:

$$L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w}) \qquad \text{and} \qquad L(a\mathbf{v}) = aL(\mathbf{v}). \tag{2.6}$$

It follows that L relates linear combinations:

$$L(C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + C_3\mathbf{v}_3) = C_1L(\mathbf{v}_1) + C_2L(\mathbf{v}_2) + C_3L(\mathbf{v}_3). \tag{2.7}$$

This property of linearity is very special. It is a standard algebra mistake to apply it to functions like the square root function and sin and cos etc. for which it does not hold. On the other hand, these should be familiar properties from calculus. The operator D associating to a differentiable function f its derivative Df is our most important example of a linear operator.

From a linear operator we get an important example of a subspace. The set of vectors $\mathbf{v}$ such that $L(\mathbf{v}) = \mathbf{0}$, the *solution space of the homogeneous equation*, is a subspace when L is a linear operator but usually not otherwise. If $\mathbf{r}$ is not $\mathbf{0}$ then the solution space of $L(\mathbf{v}) = \mathbf{r}$ is not a subspace. But if any particular solution $\mathbf{v}_p$ has been found, so that $L(\mathbf{v}_p) = \mathbf{r}$, then all of the solutions are of the form $\mathbf{v}_p + \mathbf{w}$ where $\mathbf{w}$ is a solution of the homogeneous equation. This is often written $\mathbf{v}_g = \mathbf{v}_p + \mathbf{v}_h$. That is, the general solution is the sum of a particular solution and the general solution of the homogeneous equation.

A subspace is called *invariant* for a linear operator L mapping the vector space to itself when L maps the subspace into itself. The most important example occurs when the subspace consists of multiples of a single nonzero vector $\mathbf{v}$. This subspace is invariant when $L(\mathbf{v}) = r\mathbf{v}$ for some scalar r. In that case, $\mathbf{v}$ is called an *eigenvector* for L with *eigenvalue* r.

Exercises 2.2. (a) Show that the set of linear combinations $C_1 \cos + C_2 \sin$ is an invariant subspace for D.

(b) Show that the set of polynomials of degree at most 3 is a subspace and that it is invariant for D.

(c) Given a real number r compute an eigenvector for D with eigenvalue r.

3 Matrices

An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns. That is, it is just a list of $m \cdot n$ numbers but listed in rectangular form instead of all in a line. However, this shape is irrelevant as far as vector addition and scalar multiplication are concerned. The zero $m \times n$ matrix $\mathbf{0}$ has a zero in each place. A matrix is called *a square matrix* if $m = n$.

However, there is a multiplication between matrices which seems a bit odd until you get used to it by seeing its applications. If A is an $m \times p$ matrix and B is a $p \times q$ matrix then the product $C = AB$ is defined and is an $m \times q$ matrix. Think of A as cut up into m rows, each of length p, and B as cut up into q columns, each of length p. C is constructed using the following pattern (in this picture A is 3×4, and B is 4×2 so that C is 3×2).

$$\begin{array}{ccccc} & & \begin{pmatrix} | & | \\ | & | \\ | & | \\ | & | \end{pmatrix} & = & B \\ A \; = & \begin{pmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{pmatrix} & \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} & = & C. \end{array} \tag{3.1}$$

In particular, the product of two $n \times n$ matrices is defined and yields an $n \times n$ matrix. The associative law $(AB)C = A(BC)$ and the distributive laws $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$ both hold and so we use them rather automatically. However, the commutative law $AB = BC$ is not true except in certain special cases and so you have to be careful about the side on which you multiply.

In this course we will be dealing with 2×2 matrices and their 1×2 *row vectors* and 2×1 *column vectors*.

The 2×2 *identity matrix* is $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. That is, it has 1's on the diagonal and zeroes off the diagonal. For any 2×2 matrix A, we have $IA = A = AI$. That is, I behaves like the number 1 with respect to matrix multiplication. In general, $(rI)A = rA = A(rI)$ for any scalar r.

The inverse of a 2×2 matrix A, denoted A^{-1} when it exists, is a matrix which *cancels* A by multiplication. That is $A^{-1}A = I = AA^{-1}$.

The determinant of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$ and is denoted $det(A)$. For example, $det(I) = 1$. It turns out that $det(AB) = det(A)det(B)$. This property implies that if A has an inverse matrix then $det(A)det(A^{-1}) = det(I) = 1$. So $det(A)$ is not zero. In fact, its reciprocal is $det(A^{-1})$.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ define $\widehat{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. That is, you interchange the diagonal elements and leave the other two elements in place but with the sign reversed. You should memorize this pattern as it is the reason that 2×2 matrices are much easier to deal with than larger ones. We will repeatedly use the following important identity:

$$A \cdot \widehat{A} \;=\; det(A)I \;=\; \widehat{A}A. \tag{3.2}$$

Thus, if $det(A)$ is not zero, then A has an inverse and its inverse is

$$A^{-1} \;=\; \frac{1}{det(A)} \cdot \widehat{A}. \tag{3.3}$$

If $det(A) = 0$ then A does not have an inverse and $A\widehat{A} = \mathbf{0} = \widehat{A}A$.

The *trace* of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $a + d$ and is denoted $tr(A)$.

The *transpose* of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A' = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ obtained be reversing the rows and columns. In the 2×2 case we just interchange the off-diagonal elements. Check that $det(A') = det(A)$ and that

$$A'A \;=\; \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}. \tag{3.4}$$

Taking the transpose reverses the order of multiplication in a product. That is, $(AB)' = B'A'$.

Using a 2×2 matrix A we can define two linear mappings on $\mathbb{R}^2$. Define $R_A(X) = AX$ with X a 2×1 column vector. Define $L_A(Z) = ZA$ with Z a 1×2 row vector.

The system of two linear equations in two unknowns

$$\begin{aligned} ax \;+\; by \;&=\; r, \\ cx \;+\; dy \;&=\; s. \end{aligned} \tag{3.5}$$

Can be written as a matrix equation $L_A(X) = AX = S$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the *coefficient matrix*, and with column vectors $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $S = \begin{pmatrix} r \\ s \end{pmatrix}$. When the determinant is not zero then the unique solution is $X = A^{-1}S$.

Exercises 3.1. By using the formula for the inverse, derive *Cramer's Rule*

$$x = \frac{1}{det(A)} det \begin{pmatrix} r & b \\ s & d \end{pmatrix}, \qquad y = \frac{1}{det(A)} det \begin{pmatrix} a & r \\ c & s \end{pmatrix} \tag{3.6}$$

If $det(A) = 0$ and $S = \mathbf{0}$, then the homogeneous system $AX = \mathbf{0}$ has infinitely many solutions. $X = \mathbf{0}$ is always a solution for the homogeneous system, but (3.2) implies that $A\widehat{A} = \mathbf{0}$. This means that the columns of $\widehat{A}$, $X = \begin{pmatrix} d \\ -c \end{pmatrix}$ and $X = \begin{pmatrix} -b \\ a \end{pmatrix}$ are solutions as well. At least one of these is not zero, unless A itself is $\mathbf{0}$ and in that case every vector X is a solution.

4 Linear Independence and the Wronskian

A list of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is called *linearly dependent* if one of them can be written as a linear combination of the others. For example, consider the functions $\{\cos^2(x), \sin^2(x), \cos(2x)\}$. Since $\cos(2x) = \cos^2(x) - \sin^2(x)$ it follows that this list of three functions is linearly dependent. When a list is not linearly dependent we call it *linearly independent*.

For our applications we will be looking just at pairs of vectors, lists with $k = 2$. In that case, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent when one is a multiple of the other. This means $\mathbf{v}_1 = C\mathbf{v}_2$ in which case, $\frac{1}{C}\mathbf{v}_1 = \mathbf{v}_2$ and each is a multiple of the other. This is true <u>except</u> in the annoying case when one of the vectors is $\mathbf{0}$. For any vector $\mathbf{v}$, we have $\mathbf{0} = 0 \cdot \mathbf{v}$ and so $\{\mathbf{0}, \mathbf{v}\}$ is a linearly dependent list for any vector $\mathbf{v}$.

Thus, a pair of vectors is linearly independent when the vectors are "really different".

For two differentiable, real-valued functions u and v defined for x in some interval of $\mathbb{R}$, we define the *Wronskian* $W(u, v)$ to be a new real valued function given by:

$$W(u, v) \ = \ det \begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \ = \ uv' \ - \ vu'. \tag{4.1}$$

If $v = Cu$ then the Wronskian is zero everywhere. The converse is <u>almost</u> true. Suppose the Wronskian is identically zero. We divide by u^2 to get

$$0 \ = \ \frac{W}{u^2} \ = \ \frac{uv' - vu'}{u^2} \ = \ (\frac{v}{u})' \tag{4.2}$$

by the Quotient Rule. But since the derivative is identically zero, the fraction $\frac{v}{u}$ is a constant and so $v = Cu$. Before looking at the next lines, consider this: We divided by u^2. When might that be a problem?

Now look at $u(x) = x^3$ and $v(x) = |x^3|$. When x is positive, $u(x) = v(x)$ and when x is negative $u(x) = -v(x)$. The Wronskian is identically zero, but the pair $\{u, v\}$ is linearly independent.

The problem was division by u^2 at a point x where $u(x) = 0$. On any interval where neither u nor v is zero, the ratio is indeed a constant and the pair is linearly dependent.

For our applications we will consider two vectors (a, b) and (c, d) in $\mathbb{R}^2$ which we put together as the two rows of the 2×2 matrix A. All of the following conditions are equivalent.

- The pair $\{(a, b), (c, d)\}$ is linearly dependent.
- The determinant $det(A)$ equals zero.
- The linear system $WA = \mathbf{0}$ has a solution $W = (w_1 \;\; w_2)$ which is not zero.

First, put aside the case where $A = \mathbf{0}$ and so both vectors are $\mathbf{0}$. In that case the determinant is zero, the vectors are linearly dependent and any 1×2 matrix W solves the system. Now we assume that A is not the zero matrix.

If $(a, b) = C(c, d)$ then $det(A) = Ccd - Cdc = 0$ and $W = (1 \;\; -C)$ is a nonzero solution of the system.

If $W = (w_1 w_2)$ is a solution of the system then $w_1(a, b) = -w_2(c, d)$ and so if $w_1 \neq 0$ we can divide to write (a, b) as a constant times (c, d) and similarly if $w_2 \neq 0$. So if there is a nonzero solution Z then the vectors are linearly dependent and so the determinant is zero.

If $det(A) = 0$ then by (3.2) $\widehat{A}A = \mathbf{0}$ and so $W = (d \;\; -b)$ and $W = (-c \;\; a)$, the two rows of $\widehat{A}$ are solutions of $WA = \mathbf{0}$. Since A is not the zero matrix, at least one of these is not the zero vector.

Take careful note of this trick of using the rows of $\widehat{A}$ to solve the system $WA = \mathbf{0}$. We will later use it to avoid methods like Gaussian Elimination which are needed to solve such systems when there are more variables.

5 Complex Numbers

We represent the complex number $z = a + \mathbf{i}b$ as a vector in $\mathbb{R}^2$ with x-coordinate a and y-coordinate b. Addition and multiplication by real scalars is done just as with vectors. For multiplication $(a + \mathbf{i}b)(c + \mathbf{i}d) = (ac - bd) + \mathbf{i}(ad + bc)$. The *conjugate* $\bar{z} = a - b\mathbf{i}$ and $z\bar{z} = a^2 + b^2 = |z|^2$ which is positive unless $z = 0$. So to perform division with a nonzero denominator we rationalize: $\frac{w}{z} = \frac{w\bar{z}}{z\bar{z}}$. You should check that $\overline{zw} = \bar{z} \cdot \bar{w}$ and $\overline{z + w} = \bar{z} + \bar{w}$ for complex numbers $z = a + \mathbf{i}b$ and $w = c + \mathbf{i}d$.

Multiplication is also dealt with by using the *polar* form of the complex number. This requires a bit of review.

Recall from Math 203 the three important Maclaurin series:

$$\begin{aligned} e^t &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + ... \\ \cos(t) &= 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + ... \\ \sin(t) &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + ... \end{aligned} \tag{5.1}$$

The exponential function is the first important example of a function f on $\mathbb{R}$ which is neither *even* with $f(-t) = f(t)$ nor *odd* with $f(-t) = -f(t)$ nor a mixture of even and odd functions in an obvious way like a polynomial. It turns out that any function f on $\mathbb{R}$ can be written as the sum of an even and an odd function by writing

$$f(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2}. \tag{5.2}$$

You have already seen this for $f(t) = e^t$

$$\begin{aligned} e^{-t} &= 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + ... \\ \cosh(t) = \frac{e^t + e^{-t}}{2} &= 1 + \frac{t^2}{2} + \frac{t^4}{4!} + \frac{t^6}{6!} + ... \\ \sinh(t) = \frac{e^t - e^{-t}}{2} &= t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + ... \end{aligned} \tag{5.3}$$

Now we use the same trick but with $e^{\mathbf{i}t}$

$$\begin{aligned}
e^{\mathbf{i}t} &= 1 + \mathbf{i}t - \frac{t^2}{2} - \mathbf{i}\frac{t^3}{3!} + \frac{t^4}{4!} + \mathbf{i}\frac{t^5}{5!} - \frac{t^6}{6!} - \mathbf{i}\frac{t^7}{7!} + ... \\
e^{-\mathbf{i}t} &= 1 - \mathbf{i}t - \frac{t^2}{2} + \mathbf{i}\frac{t^3}{3!} + \frac{t^4}{4!} - \mathbf{i}\frac{t^5}{5!} - \frac{t^6}{6!} + \mathbf{i}\frac{t^7}{7!} + ... \\
\frac{e^{\mathbf{i}t} + e^{-\mathbf{i}t}}{2} &= 1 \quad - \frac{t^2}{2} \quad + \frac{t^4}{4!} \quad - \frac{t^6}{6!} + ... \\
\frac{e^{\mathbf{i}t} - e^{-\mathbf{i}t}}{2} &= \mathbf{i}[t \quad - \frac{t^3}{3!} \quad + \frac{t^5}{5!} \quad - \frac{t^7}{7!} + ...]
\end{aligned} \tag{5.4}$$

So the even part of $e^{\mathbf{i}t}$ is $\cos(t)$ and the odd part is $\mathbf{i}\sin(t)$. Adding them we obtain $Euler'sIdentity$ and its conjugate version.

$$\begin{aligned}
e^{\mathbf{i}t} &= \cos(t) + \mathbf{i}\sin(t) \\
e^{-\mathbf{i}t} &= \cos(t) - \mathbf{i}\sin(t)
\end{aligned} \tag{5.5}$$

Substituting $t = \pi$ we obtain $e^{\mathbf{i}\pi} = -1$ and so $e^{\mathbf{i}\pi} + 1 = 0$.

Now we start with a complex number z in rectangular form $z = x + \mathbf{i}y$ and convert to polar coordinates with $x = r\cos(\theta), y = r\sin(\theta)$ so that $r^2 = x^2 + y^2 = z\bar{z}$. The length r is called the *magnitude* of z. The angle θ is called the *argument* of z. We obtain

$$z = x + \mathbf{i}y = r(\cos(\theta) + \mathbf{i}\sin(\theta)) = re^{\mathbf{i}\theta}. \tag{5.6}$$

If $z = re^{\mathbf{i}\theta}$ and $w = ae^{\mathbf{i}\phi}$ then, by using properties of the exponential, we see that $zw = rae^{\mathbf{i}(\theta+\phi)}$. That is, the magnitudes multiply and the arguments add. In particular, for a whole number n we have that $z^n = r^n e^{\mathbf{i}n\theta}$.

DeMoivre's Theorem describes the solutions of the equation $z^n = a$ which has n distinct solutions when a is nonzero. We describe these solutions when a is a nonzero real number.

Notice that $re^{\mathbf{i}\theta} = re^{\mathbf{i}(\theta+2\pi)}$. When we raise this equation to the power n the angles are replaced by $n\theta$ and $n\theta + n2\pi$ and so they differ by a multiple of π. But when we divide by n we get different angles. So if $a > 0$, then $a = e^{\mathbf{i}0}$ and $-a = e^{\mathbf{i}\pi}$.

$$\begin{aligned}
z^n &= a \quad : \quad z = a^{1/n} \cdot \{e^{\mathbf{i}0}, e^{\mathbf{i}2\pi/n}, e^{\mathbf{i}4\pi/n}, ..., e^{\mathbf{i}(n-1)2\pi/n}\} \\
z^n &= -a \quad : \quad z = a^{1/n} \cdot \{e^{\mathbf{i}\pi/n}, e^{\mathbf{i}(\pi+2\pi)/n}, e^{\mathbf{i}(\pi+4\pi)/n}, ..., e^{\mathbf{i}(\pi+(n-1)2\pi)/n}\}
\end{aligned} \tag{5.7}$$

Another way of handling complex numbers is to associate to the complex number $z = x + \mathbf{i}y$ the 2×2 real matrix $A(z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$. If $\xi = \rho + \mathbf{i}\omega$ so that $A(\xi) = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}$ and $\xi z = u + \mathbf{i}v$, we have that

$$A(\xi z) \;=\; A(\xi)A(z) \qquad \text{and} \qquad \begin{pmatrix} u \\ v \end{pmatrix} \;=\; \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5.8}$$

Exercises 5.1. Verify the two equations of (5.8).

6 Linear Systems of Differential Equations - Special Cases

Just as a system of linear equations can be written as a single matrix equation $AX = S$ with A an $n \times n$ matrix and with X and S $n \times 1$ column vectors, so we can write a system of n linear differential equations as $\frac{dX}{dt} = AX$. This looks just like exponential growth and so the solution should be $X = e^{tA}X_0$ with X_0 the $n \times 1$ vector describing the initial position. That is, $X(0) = X_0$. Notice that e^{tA} will be an $n \times n$ matrix and so we have to multiply the $n \times 1$ matrix X_0 on the right.

The question is: what does that exponential mean? The first thought would be to apply the exponential function to every entry of the matrix tA, but that doesn't work. Instead, you define it by analogy with the expansion of e^t given by (5.1):

$$e^{tA} \;=\; I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \frac{t^5}{5!}A^5 + ... \tag{6.1}$$

Here I is the $n \times n$ identity matrix with 1's on the diagonal and 0's otherwise, just as in the 2×2 case. The powers $A, A^2, ...$ are just obtained by repeatedly multiplying A times itself. Even in the 2×2, which is all we will deal with, computing A^5 for example, just from the definition is pretty nasty job. So some tricks are needed to make use of this general solution. In this section we will compute some special cases and in the next we will introduce the tricks which handle the general 2×2 case.

Before we leave the general case, we look at two ideas which describe how we will proceed.

The first idea is to change coordinates in a way that makes the problem easier.

Think back to inclined plane problems in your first physics course. While our vectors are naturally given in terms of horizontal and vertical coordinates, the first thing you do in the plane problems is write all of the vectors in terms of a component perpendicular to the plane and a component parallel to the plane.

Here a linear change of coordinates replaces X by $Y = QX$ where Q is a constant matrix which has an inverse so that we can recover $X = Q^{-1}Y$. Because Q is a constant and the derivative is a linear operator we have

$$\frac{dY}{dt} \ = \ \frac{dQX}{dt} \ = \ Q\frac{dX}{dt} \ = \ QAX \ = \ QAQ^{-1}QX \ = \ QAQ^{-1}Y. \qquad (6.2)$$

In the Y system the coefficient matrix A has been replaced by the *similar* matrix QAQ^{-1}. The trick will be to choose Q so that QAQ^{-1} has a simple form.

The other idea is to notice that $Z(t) = e^{tA}$ is a path in the space of $n \times n$ matrices with $Z(0) = I$. That is, it is the unique solution of the initial value problem in the space of matrices:

$$\frac{dZ}{dt} \ = \ AZ \qquad \text{with} \qquad Z(0) \ = \ I. \qquad (6.3)$$

This is called the *fundamental solution* for the system. Then if we want the solution of the initial value problem in $\mathbb{R}^n$ given by $\frac{dX}{dt} = AX$ with $X(0) = X_0$ we just use $X(t) = Z(t)X_0$. Notice that if Q is an invertible matrix that we want to use to change coordinates then

$$\frac{dQZQ^{-1}}{dt} \ = \ (QAQ^{-1})(QZQ^{-1}) \qquad \text{with} \qquad QZQ^{-1}(0) \ = \ QIQ^{-1} \ = \ I. \qquad (6.4)$$

That is, the fundamental solution for the system with A replaced by the similar matrix QAQ^{-1} is obtained by replacing $Z(t)$ by the same similarity transformation at each time t.

In the next section we will explain the labels used for the three cases we now consider. As you might anticipate from our previous work with second order, linear, homogeneous equations with constant coefficients, the labels refer to the roots of an appropriate *characteristic equation.*

Case 1: Two Real Roots r_1, r_2 **:** The special system here is

$$\begin{cases} \frac{dx}{dt} = r_1 x, \\ \frac{dy}{dt} = r_2 y. \end{cases} \qquad A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \tag{6.5}$$

In this case, the two variables x and y are said to be *uncoupled.* We solve the x and y equations separately to get $x = x_0 e^{r_1 t}, \;\; y = y_0 e^{r_2 t}$. The fundamental solution is given by

$$Z(t) = \begin{pmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{pmatrix}. \tag{6.6}$$

Case 2: Repeated Real Root r **:** The special system this time is

$$\begin{cases} \frac{dx}{dt} = rx, \\ \frac{dy}{dt} = x + ry. \end{cases} \qquad A = \begin{pmatrix} r & 0 \\ 1 & r \end{pmatrix}. \tag{6.7}$$

Here we can solve the x equation to get $x = x_0 e^{rt}$ and we can substitute this into the y equation to get $\frac{dy}{dt} = x_0 e^{rt} + \; ry$. This is a simple example of the first order linear equations studied in Section 2.1 of B & D with integrating factor e^{-rt} and solution $y = t x_0 e^{rt} + y_0 e^{rt}$. The fundamental solution is

$$Z(t) = \begin{pmatrix} e^{rt} & 0 \\ te^{rt} & e^{rt} \end{pmatrix} = e^{rt} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}. \tag{6.8}$$

Case 3: Complex Conjugate Roots $\rho \pm \mathbf{i}\omega$ **:** In this case the special system is

$$\begin{cases} \frac{dx}{dt} = \rho x - \omega y, \\ \frac{dy}{dt} = \omega x + \rho y. \end{cases} \qquad A = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}. \tag{6.9}$$

We switch to complex coordinates. Let $z = x + \mathbf{i}y$ and $\xi = \rho + \mathbf{i}\omega$. By using the second equation of (5.8) we see that the system becomes $\frac{dz}{dt} = \xi z$. Using the complex exponential we get the solution $z = e^{\xi t} z_0$ where $z_0 = x_0 + \mathbf{i} y_0$. Remember that

$$e^{\xi t} \ = \ e^{\rho t + \mathbf{i}\omega t} \ = \ e^{\rho t} e^{\mathbf{i}\omega t} \ = \ e^{\rho t}[\cos(\omega t) + \mathbf{i}\sin(\omega t)]. \tag{6.10}$$

Now we use (5.8) again to go back from complex numbers to vectors in $\mathbb{R}^2$. Multiplying the complex number $x_0 + \mathbf{i} y_0$ by the complex number $e^{\rho t}[\cos(\omega t) + \mathbf{i}\sin(\omega t)]$ becomes multiplication of the 2×1 column vector $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ by the fundamental matrix

$$Z(t) \ = \ e^{\rho t} \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}. \tag{6.11}$$

In this case, it is useful to consider the system in polar coordinates. In complex language $z = re^{\mathbf{i}\theta}$ with $r^2 = z\bar{z}$. Now we start with $\frac{dz}{dt} = \xi z$ from which we get $\frac{d\bar{z}}{dt} = \bar{\xi}\bar{z}$. (Remember that $\overline{\xi z} = \bar{\xi}\bar{z}$). So we get, by the Product Rule,

$$\begin{aligned} 2r\frac{dr}{dt} \ &= \ \frac{dr^2}{dt} \ = \ \frac{dz\bar{z}}{dt} \\ = \ \frac{dz}{dt}\bar{z} + z\frac{d\bar{z}}{dt} \ &= \ (\xi + \bar{\xi})z\bar{z} \ = \ 2\rho r^2. \end{aligned} \tag{6.12}$$

and so $\frac{dr}{dt} \ = \ \rho r$.

Next, we have, by the Product Rule again and the Chain Rule,

$$\begin{aligned} (\rho + \mathbf{i}\omega)re^{\mathbf{i}\theta} \ &= \ \xi z \ = \ \frac{dz}{dt} \ = \ \frac{dre^{\mathbf{i}\theta}}{dt} \\ = \ \frac{dr}{dt}e^{\mathbf{i}\theta} + \mathbf{i}re^{\mathbf{i}\theta}\frac{d\theta}{dt} \ &= \ (\rho + \mathbf{i}\frac{d\theta}{dt})re^{\mathbf{i}\theta}. \end{aligned} \tag{6.13}$$

This implies $\frac{d\theta}{dt} \ = \ \omega$.

So we obtain the system in polar coordinates:

$$\frac{dr}{dt} \ = \ \rho\, r \qquad \text{and} \qquad \frac{d\theta}{dt} \ = \ \omega. \tag{6.14}$$

We conclude by noting that the second order, linear, homogeneous equation with constant coefficients $A\frac{d^2x}{dt^2} \ + \ B\frac{dx}{dt} \ + \ Cx \ = \ 0$ can be rewritten in the form of a 2×2 system. Let $v = \frac{dx}{dt}$ to get the system

$$\begin{cases} \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= (-C/A)x \ + \ (-B/A)v. \end{cases} \qquad A \ = \ \begin{pmatrix} 0 & 1 \\ (-C/A) & (-B/A) \end{pmatrix}. \tag{6.15}$$

7 Eigenvalues and Eigenvectors

A (left) eigenvector with eigenvalue r for a 2×2 matrix A is a nonzero 1×2 row vector W such that $WA \ = \ rW$. If we have an eigenvector then we get the eigenvalue by seeing what multiple of W is WA. However, in practise we find the eigenvalue first and then compute its eigenvector. Notice that we speak of the eigenvector although we shouldn't because any nonzero multiple of W is an eigenvector with the same eigenvalue.

The vector W is an eigenvector with eigenvalue r exactly when it is a nonzero solution of $W(A - rI) = \mathbf{0}$. We saw in Section 4 that such a nonzero solution can be found only when the determinant $det(A - rI) = 0$ in which case we can use $\widehat{A - rI}$ to solve the system. So we first find out for which real or complex numbers r the determinant $det(A - rI)$ is zero and then for each compute an eigenvector. When we expand this determinant we get a quadratic equation in the unknown r. With $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$det(A - rI) \ = \ det(\begin{pmatrix} a - r & b \\ c & d - r \end{pmatrix} \ = \ r^2 \ - \ tr(A)r \ + \ det(A). \tag{7.1}$$

where the trace $tr(A) = a + d$ and the determinant $det(A) = ad - bc$. The associated quadratic equation $r^2 - tr(A)r + det(A) \ = \ 0$ is called the *characteristic equation* of the matrix A. We get the roots r_+ and r_- by applying the Quadratic Formula.

$$\begin{gathered} r_{\pm} \ = \ \frac{1}{2}[\ tr(A) \ \pm \ \sqrt{tr(A)^2 \ - \ 4det(A)} \ \ ], \qquad \text{and so} \\ r_+ \ + \ r_- \ = \ tr(A) \qquad \text{and} \qquad r_+ \cdot r_- \ = \ det(A). \end{gathered} \tag{7.2}$$

The expression inside the square root is the *discriminant*, $disc(A)$, which will account for our different cases.

$$disc(A) \;=\; tr(A)^2 \;-\; 4det(A) \;=\; (a+d)^2 \;-\; 4(ad-bc) \;=\; (a-d)^2 \;+\; 4bc. \tag{7.3}$$

Case 1: $disc(A) > 0,\ 4det(A) < tr(A)^2$**, Two Real Roots** r_+, r_- **.**

We know that $det(A - r_+I) = 0$ and so the rows $(d - r_+ \quad -b)$ and $(-c \quad a - r_+)$ of $\widehat{A - r_+I}$ are solutions of $W(A - r_+I) = 0$. Notice that $disc(A) \neq 0$ implies that if either b or c equals 0 then $a \neq d$. So at least one of these rows is nonzero (usually both). Choose a nonzero one as the eigenvector W_+ with eigenvalue r_+. The two rows are linearly dependent and so up to nonzero multiple we get only one eigenvector whichever choice we use.

For r_- we do the same thing use either $(d - r_- \quad -b)$ or $(-c \quad a - r_-)$, whichever is nonzero, as a choice for the eigenvector W_- with eigenvector r_-.

Exercises 7.1. Using the facts that $r_+ \neq r_-$ and that neither W_+ nor W_- is the zero vector, show that the pair $\{W_+, W_-\}$ is linearly independent. (Hint: Assume $W_- = CW_+$, multiply by A and derive a contradiction.)

Define Q as the 2×2 matrix with rows W_+ and W_-. Because these are linearly independent, $det(Q)$ is not zero and so Q has an inverse Q^{-1}. From the definitions of these vectors we have

$$QA \;=\; \begin{pmatrix} r_+ & 0 \\ 0 & r_- \end{pmatrix} Q \qquad \text{and so} \qquad QAQ^{-1} \;=\; \begin{pmatrix} r_+ & 0 \\ 0 & r_- \end{pmatrix} \tag{7.4}$$

Case 2: $disc(A) = 0,\ 4det(A) = tr(A)^2$**, One Real Root,** r**, Repeated.**

Since the discriminant $(d-a)^2 - 4bc = 0$ we see that

$$a \;=\; d \qquad \Longleftrightarrow \qquad b = 0 \;\text{ or }\; c = 0. \tag{7.5}$$

Also, the root r is $\frac{1}{2}tr(A) \;=\; \frac{1}{2}(a+d)$ and so

$$A - rI \;=\; \begin{pmatrix} (a-d)/2 & b \\ c & (d-a)/2 \end{pmatrix}. \tag{7.6}$$

If $A - rI$ is the zero matrix then $A = rI = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$. That is, A is already in the special form we want and so we make no change. Or you can think that we use $Q = I$.

From now on assume that $A - rI$ is not zero. So we use as our eigenvector W whichever of the two rows of $\widehat{A - rI}$ are nonzero. That is, we use either $W^{(1)} = ((d-a)/2 \quad -b)$ or $W^{(2)} = (-c \quad (a-d)/2)$. Whatever the choice we get only one eigenvector (up to nonzero multiple) and we cannot build Q using eigenvectors alone. Instead we use a different sort of vector. Let $U^{(1)} = (-1\ 0)$ and $U^{(2)} = (0\ -1)$. Observe that

$$\begin{aligned} U^{(1)}(A - rI) \;&=\; W^{(1)} \qquad \text{and} \qquad U^{(2)}(A - rI) \;=\; W^{(2)}, \qquad \text{and so} \\ U^{(1)}A \;&=\; W^{(1)} + rU^{(1)} \qquad \text{and} \qquad U^{(2)}A \;=\; W^{(2)} + rU^{(2)}. \end{aligned} \tag{7.7}$$

Let Q be the 2×2 matrix with rows $W^{(1)}$ and $U^{(1)}$ which has determinant $-b$. If $b = 0$ and so $W^{(1)} = 0$ we use the rows $W^{(2)}$ and $U^{(2)}$ to obtain Q with determinant c. From the definitions of these vectors we have

$$QA \;=\; \begin{pmatrix} r & 0 \\ 1 & r \end{pmatrix} Q \qquad \text{and so} \qquad QAQ^{-1} \;=\; \begin{pmatrix} r & 0 \\ 1 & r \end{pmatrix} \tag{7.8}$$

Case 3: $disc(A) < 0,\ 4det(A) > tr(A)^2$, **Complex Conjugate Pair** $\rho \pm \mathbf{i}\omega$ **.**

Since $disc(A) = (a-d)^2 + 4bc < 0,\ 4bc < -(d-a)^2$ and so neither b nor c is zero. By (7.2) $tr(A) = 2\rho$ and $det(A) = \rho^2 + \omega^2$. The real part ρ might be zero but $\omega = \frac{1}{2}\sqrt{|disc(A)|} \neq 0$.

With $\xi \;=\; \rho + \mathbf{i}\omega$ the complex matrix $A - \xi I$ has determinant zero and so the first row of $\widehat{A - \xi I},\ Z = (d - \xi \quad -b)$ is an eigenvector of A with eigenvalue ξ. That is, with $Z = (z_1\ \ z_2)$ we have $ZA = \xi Z$. Since conjugation commutes with multiplication and addition we have $\bar{Z}A = \bar{\xi}\bar{Z}$. That is, $\bar{Z}$ is an eigenvector with eigenvalue $\bar{\xi}$. Here $z_1 = d - \rho \ -\ \mathbf{i}\omega,\ z_2 = -b$. So if we let

$$Q \;=\; \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \;=\; \begin{pmatrix} d-\rho & -b \\ -\omega & 0 \end{pmatrix} \tag{7.9}$$

then Q has determinant $-b\omega$ which is not zero. Thus, the two columns of Q are the 2×1 matrices associated with the complex numbers z_1 and z_2.

We check that in this case

$$QA \;=\; \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} Q \qquad \text{and so} \qquad QAQ^{-1} \;=\; \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}. \tag{7.10}$$

First notice that since A is real (no imaginary part), the real and imaginary parts of ZA are given by $(x_1 \;\; x_2)A$ and $(y_1 \;\; y_2)A$, respectively and so they are the two rows of ZA.

Next, recall from (5.8) that the real and imaginary parts of ξz_1 are given by $\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and similarly for ξz_2. So we see that the real and imaginary parts of $\xi(z_1 \;\; z_2)$ are the first and second rows of $\begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} Q$.

This verifies the first equation of (7.10) and the second follows, as usual, by multiplying by Q^{-1} on the right.

Thus, in each of these cases we have found the invertible matrix Q which converts the matrix A to the associated special form considered in the previous section.

Exercises 7.2. Check that each special form has the expected eigenvalues. That is,

$$A \;=\; \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \qquad \text{has eigenvalues} \quad r_1, r_2.$$

$$A \;=\; \begin{pmatrix} r & 0 \\ 1 & r \end{pmatrix} \qquad \text{has eigenvalue} \quad r \text{ and } disc(A) = 0, \tag{7.11}$$

$$A \;=\; \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \qquad \text{has eigenvalues} \quad \rho \pm \mathbf{i}\omega.$$

8 Bifurcations

The behavior of the linear differential equation $\frac{dx}{dt} = rx$ depends only on the sign of r. This is exponential growth when $r > 0$ and exponential decay when $r < 0$. The equilibrium at 0 is a attractor when $r < 0$ and is a repellor when $r > 0$.

Exercises 8.1. Show that if r_1 and r_2 are either both positive or both negative then on the positive real numbers the change of variables $y = x^{r_2/r_1}$ converts the equation $\frac{dx}{dt} = r_1 x$ to $\frac{dy}{dt} = r_2 y$.

Think of the parameter r as changing, but very slowly compared to the time-scale of the equation. All the while that r remains negative the particle moving according to the equation moves rapidly toward 0. Suddenly, when $r = 0$ the entire universe changes at once. Everything stops. As long as $r = 0$ every point is an equilibrium. Then once r is positive 0 becomes a repellor and every point everywhere moves away from zero.

The parameter value $r = 0$ is called a *bifurcation value* separating two regimes of very different behavior.

Now look at the nonlinear case, with $\frac{dx}{dt} = rx - x^3$. Using the graphical methods of Section 1, you can see that as long as $r < 0$ there is a unique equilibrium at $x = 0$ and every point moves in toward 0. This time even when $r = 0$ the equilibrium at 0 is a global attractor. Again $r = 0$ is a bifurcation value but now the behavior changes to something more interesting.

Exercises 8.2. Show that if $r > 0$ then $\frac{dx}{dt} = rx - x^3$ has a repelling equilibrium at $x = 0$ and attracting equilibria at $x = \pm\sqrt{r}$.

The behavior near $x = 0$ is like that of the linear system $\frac{dx}{dt} = rx$ when r is not zero, but the points near infinity still move in toward 0 as they did before. Separating this old attracting regime and the new repelling regime are the new equilibria.

Things can become even more interesting for systems of equations.

Exercises 8.3. With $\xi = \rho + \mathbf{i}\omega$, show that the complex equation $\frac{dz}{dt} = \xi z - r^2 z$ (with $r^2 = z\bar{z}$) can be written in rectangular coordinates, and in polar coordinates as

$$\begin{cases} \frac{dx}{dt} &= \rho x - \omega y - x(x^2 + y^2), \\ \frac{dy}{dt} &= \omega x + \rho y - y(x^2 + y^2). \end{cases} \qquad \text{and} \qquad \begin{cases} \frac{dr}{dt} &= \rho r - r^3, \\ \frac{d\theta}{dt} &= \omega. \end{cases} \tag{8.1}$$

By looking at the system in polar coordinates, we see again that the origin $(0,0)$ is a globally attracting equilibrium for $\rho \leq 0$. Again when ρ becomes positive, the equilibrium becomes a repellor with nearby behavior like that of the linear system. Again the points at infinity still move in toward the origin. But now the orbits all spiral with with angular velocity ω. $r = \sqrt{\rho}$ at which $\frac{dr}{dt} = 0$ is not an equilibrium, but instead an attracting circle. This appearance of a *limit cycle* was due to the bifurcation in which there was a sign change in the real part ρ of a complex conjugate pair of eigenvalues. It is an example of a *Hopf bifurcation*.

Syllabus and Homework

In the syllabus below we write BD for the Boyce and DiPrima Book and EA for these notes. The BD homework assignments refer to the 10^{th} Edition [with the same problems in the 9^{th} Edition given in brackets].

1. Introduction to First Order Equations: EA Section 1 and BD Section 2.2, Separable and Homogeneous equations, HW: EA ex 1.1, BD p. 48/ 3-17odd [pp. 47-48/ 3-17odd] and pp. 50-51/ 31-37odd [p. 50/ 31-37odd].

2. Vectors and Linear Equations: EA Section 2 and BD Section 2.1, Linear Equations. HW: EA ex 2.1, 2.2 BD p. 40/ 7-10, 11-15odd [p. 39/ 7-10, 11-15odd].

3. Potentials and Exact Equations: BD Section 2.6, HW: BD p. 101/ 1, 3, 4, 7-13odd, 17, 18 [pp. 99-100/ 1, 3, 4, 7-13odd, 17, 18 differential notation].

4. Existence and Uniqueness, Linear vs Nonlinear Equations: BD Section 2.4, HW: Miscellaneous pp. 133-134/ 1-14, 28, 29 [pp. 132-133/ 1-14, 28, 29].

5. Modeling: BD Section 2.3, HW: BD pp. 60-63/ 1-4, 7-10, 16 [pp. 59-62/ 1-4, 7-10, 16].

6. Reduction of Order Problems: HW: BD pp. 135-136/ 36, 37, 41, 42, 43, 45, 48, 49 [pp. 134-135/ 36, 37, 41, 42, 43, 45, 48, 49].

7. Introduction to Second Order Linear Equations and Matrices: EA

Section 3 (see also BD Section 7.2), BD Section 3.1, HW: EA ex 3.1, BD p. 144/ 1-11odd, 16 [p. 144/ 1-11odd, 16], and pp. 376-377/2, 10, 11 [pp. 371-372/2, 10, 11].

8. Linear Independence, Fundamental Solutions and the Wronskian: EA Section 4, BD Section 3.2 HW: BD pp. 155-157/ 1, 3, 5, 14, 16, 38, 39 [pp. 155-156/ 1, 3, 5, 14, 16, 38, 39].

9. Complex Roots: EA Section 5, BD Section 3.3, HW: EA ex. 5.1, BD p. 164/ 1-11odd, 17, 19 [p. 163/ 1-11odd, 17, 19].

10. Repeated Roots; Reduction of Order: BD Section 3.4, HW: BD pp. 172-174/ 1-11odd; 23, 25, 28 [pp. 171-173/ 1-11odd; 23, 25, 28].

11. Euler Equations: BD Section 3.3, p. 166, Section 3.4, p.175 , HW: BD p. 166/ 35-41 odd, p. 175/41-45 odd [p. 165/ 35-41 odd, p. 174/41-45 odd].

12. Undetermined Coefficients: BD Section 3.5, HW: BD p. 184/ 5-11odd, 12, 15, 17, and 21-25 (Y(t) alone) [p. 183/ 3-9odd, 10, 13, 15, and 19-23 (Y(t) alone)].

13. Variation of Parameters: BD Section 3.6, HW: BD p. 190/ 3, 5, 7, 9, 10, 13, 14 [p. 189/ 3, 5, 7, 9, 10, 13, 14].

14. Higher Order Linear Equations with Constant Coefficients: BD Sections 4.2, 4.3 p. 234/ 11-23odd [p. 232/ 11-23odd] and p. 239/ 13-18 (Y(t) alone) [p. 237/ 13-18 (Y(t) alone)].

15. Spring Problems: BD Sections 3.7, 3.8, HW: BD pp. 203-204/ 5, 6, 7, 9 p. 217/ 5, 6, 9 [p. 202/ 5, 6, 7, 9 p. 215/ 5, 6, 9].

16. Series Solutions: BD Section 5.2, HW: BD pp. 263-264/1, 2, 5-17 odd [pp. 259-260/1, 2, 5-17 odd].

17. Introduction to Linear Systems with Constant Coefficients: EA Section 6, BD Sections 7.5.

18. Eigenvalues and Eigenvectors, Fundamental Matrix Solutions: EA Section 7, BD Section 7.7, HW: EA ex. 7.1, 7.2 BD pp. 427-428/1-8, 11, 12, p. 436/1 [pp. 420-421/1-8, 11, 12, p. 428/1].

19. The Phase Plane: BD Sections 9.1, 9.3, HW: BD p. 505/1-5 [pp. 494-495/1-5].

20. Bifurcations: EA Section 8, HW: ex. 8.1, 8.2, 8.3.

21. Fourier Series: BD Section 10.2, HW: BD p. 605/ 13-18 [p. 593/ 13-18].

22. Even and Odd Functions: BD Section 10.4, HW: BD p. 620/ 15-19 [p. 608/ 15-19]

23. The Heat Equation: BD Section 10.5 HW: BD pp. 630-631/ 1-12 [pp. 618-619/ 1-12].

9 Class Information Notes

MATH 39104KL (65424) Differential Equations

Spring, 2015 Tuesday, Thursday 8:00-9:40 am NAC 5/101

Book and Notes: *Elementary Differential Equations and Boundary Value Problems* , 10^{th} Edition, by William E. Boyce and Richard C. DiPrima [The 9^{th} Edition is fine, but make sure you get the one that has "Boundary Value Problems" in the title.]

In addition, there will be the supplementary notes attached above, a pdf of which will be posted on my home page.

Grading: There will be three in class tests [if time permits, otherwise, just two] and a final exam. The final counts 40 % of the grade. You should be warned that there are no makeups. Instead the remaining work will simply be counted more heavily.

While I do not collect homework, it is important that you do the homework, preferably in advance of the class in which we will go over it.

Please attend regularly and be on time.

Office Hours: Tuesday 10:00-10:50 pm and Thursday 10:00-10:50 am. Other times by appointment.

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