(1) (8 points each) Compute the general solution for each of the differential equations, (y is a function of x in each case).

$$(a) \quad x^2y'' \; - \; xy' \; + \; 5y \quad = \quad 0, x > 0$$

Euler equation with indicial equation $r(r-1) - r + 5 = r^2 - 2r + 5$ with roots $1 \pm 2\mathbf{i}$.

$$y \;=\; C_1 x\cos(2\ln(x)) \;+\; C_2 x \sin(2\ln(x).$$

$$(b) \quad y'' \; - \; y' \; - \; 5y \quad = \quad 50x.$$

Characteristic equation of the homogeneous: $r^2 - r - 5$ with roots $\frac{1 \pm \sqrt{21}}{2}$. The test function $Y_p = Ax + B$.

$$\begin{aligned} -5 \times Y_p &= -5Ax \quad -5B \\ -1 \times Y_p' &= \qquad\qquad -A \\ 1 \times Y_p'' &= \quad 0. \end{aligned}$$

So $50x \;=\; -5Ax + (-5B - A)$. So $A = -10, B = -A/5 = 2$.

$$y \;=\; C_1 e^{(1+\sqrt{21})x/2} \;+\; C_2 e^{(1-\sqrt{21})x/2} \;-\; 10x \;+\; 2.$$

(2) (12 points) (a) State the definition of the Laplace transform and use it to compute to compute for a function $f(t) = e^{2t}$ the Laplace Transform $\mathcal{L}(f)(s)$.

$\mathcal{L}(y)(s) \;=\; \int_0^\infty \; e^{-st}y(t)\; dt$. With $y = e^{2t}$,
$\mathcal{L}(y)(s) \;=\; \int_0^\infty \; e^{-(s-2)t}\; dt = \frac{1}{s-2}$.

(b) Compute the Laplace Transform $\mathcal{L}(y)$ for the solution of the initial value problem:

$$7y'' - 3y' + 4y = 5e^{2t},$$
$$y(0) = 5 \quad \text{and} \quad y'(0) = 3.$$

$\mathcal{L}(y') = s\mathcal{L}(y) - y(0), \quad \mathcal{L}(y'') = s^2\mathcal{L}(y) - y'(0) - sy(0)$. So
$7(s^2\mathcal{L}(y) - 3 - s5) - 3(s\mathcal{L}(y) - 5) + 4\mathcal{L}(y) \;=\; \frac{5}{s-2}$.

$$\mathcal{L}(y) \;=\; [35s + 6 + \frac{5}{s-2}] \div [7s^2 - 3s + 4].$$

(3)(18 points) For the differential equation:

$$(2 - x^4)y'' - x^3y' - x^2y \quad = \quad 0$$

The point $x_0 = 0$ is an *ordinary point.* Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = 0$ and use it to compute the first four nonzero terms of the power series when $y(0) = 48$ and $y'(0) = 20$.

$$y'' = \sum 2n(n-1)a_n x^{n-2} = \sum 2(k+2)(k+1)a_{k+2}x^k.$$
$$-x^4y'' = \sum n(n-1)a_n x^{n+2} = \sum (k-2)(k-3)a_{k-2}x^k.$$
$$-x^3y' = \sum -na_n x^{n+2} = \sum -(k-2)a_{k-2}x^k.$$
$$-x^2y = \sum -a_n x^{n+2} = \sum -a_{k-2}x^k.$$

$$a_{k+2} = \frac{[(k-2)^2+1]a_{k-2}}{2(k+2)(k+1)}.$$

$a_0 = 48, a_1 = 20.$

$(k=0) \quad a_2 \;=\; 0.$

$(k=1) \quad a_3 \;=\; 0.$

$(k=2) \quad a_4 \;=\; \frac{a_0}{24} \;=\; 2.$

$(k=3) \quad a_5 \;=\; \frac{2a_1}{40} \;=\; 1.$

$$y \;=\; 48 + 20x + 2x^4 + x^5 + \ldots$$

(4) (18 points) For the differential equation:

$$(x+2)y'' - (x^2-1)y' + (x+1)y \quad = \quad 0$$

The point $x_0 = -1$ is an *ordinary point.* Compute the recursion formula for the coefficients of the power series solution centered at $x_0 = -1$ and use it to compute the first four nonzero terms of the power series when $y(-1) = -3$ and $y'(-1) = 3$.

Let $X = x + 1$ so that $x = X - 1$. The equation becomes

$$(1+X)y'' + (2X - X^2)y' + Xy.$$

$$\begin{aligned}
y'' &= \sum n(n-1)a_n X^{n-2} = \sum (k+2)(k+1)a_{k+2}X^k.\\
Xy'' &= \sum n(n-1)a_n X^{n-1} = \sum (k+1)(k)a_{k+1}X^k.\\
-X^2y' &= \sum -na_n X^{n+1} = \sum -(k-1)a_{k-1}X^k.\\
2Xy' &= \sum 2na_n X^n = \sum 2ka_k X^k.\\
Xy &= \sum a_n X^{n+1} = \sum a_{k-1}X^k.
\end{aligned}$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)} \times [-(k+1)ka_{k+1} - 2ka_k + (k-2)a_{k-1}].$$

$a_0 = -3, \qquad a_1 = 3.$

$$\begin{array}{lll}
(k=0) & a_2 & = \ 0.\\
(k=1) & a_3 & = \ [2a_2 - 2a_1 - a_0]/6 = -\frac{1}{2}\\
(k=2) & a_4 & = \ [-6a_3 - 4a_2 + 0]/12 = \frac{1}{4}.
\end{array}$$

$$y = -3 + 3(x+1) - \frac{1}{2}(x+1)^3 + \frac{1}{4}(x+1)^4 + \ldots$$

(5) (18 points) For the differential equation:

$$(2x^2 + x^3)y'' + (3x + x^2)y' - y \quad = \quad 0$$

The point $x = 0$ is a *regular singular point* . Compute the *associated Euler equation* and compute the recursion formula for the coefficients of the series solution centered at $x_0 = 0$ which is associated with the larger root.

Associated Euler equation is $2x^2y'' + 3xy' - y = 0$ with indicial equation $0 \ = \ 2r(r-1) + 3r - 1 = 2r^2 + r - 1 = (2r-1)(r+1)$ with roots $r = \frac{1}{2}, -1$.

$$\begin{aligned}
2x^2y'' &= \sum 2(n+r)(n+r-1)a_n x^{n+r} = \sum 2(k+r)(k+r-1)a_k x^{k+r}.\\
x^3y'' &= \sum (n+r)(n+r-1)a_n x^{n+r+1} = \sum (k+r-1)(k+r-2)a_{k-1}x^{k+r}.\\
3xy' &= \sum 3n(n+r)a_n x^{n+r} = \sum 3(k+r)a_k x^{k+r}\\
x^2y' &= \sum (n+r)a_n x^{n+r+1} = \sum (k+r-1)a_{k-1}x^{k+r}\\
-y &= \sum a_n x^{n+r} = \sum a_k x^{k+r}
\end{aligned}$$

$$\begin{aligned}
&[2(k+r)(k+r-1) + 3(k+r) + 1]a_k = (2(k+r)-1)((k+r)+1)a_k =\\
&\qquad -[(k+r-1)(k+r-2) + (k+r-1)]a_{k-1} = -(k+r-1)^2 a_{k-1}.
\end{aligned}$$

With $r = 1/2$, $\quad k(2k+3)a_k = -(k-\frac{1}{2})^2 a_{k-1}$ or

$$a_k = -\frac{(2k-1)^2 a_{k-1}}{4k(2k+3)}.$$

(6)(18 points) (a) Compute the Fourier series for the function $f(x) = |x|$ on the interval $[-2, 2]$. Recall that $|x| = \begin{cases} -x \text{ for } x \leq 0, \\ x \text{ for } x \geq 0. \end{cases}$

The absolute value of x is an even function and so $b_n = 0$ for all n.

$$a_0 = \frac{1}{2} \times 2 \int_0^2 x dx = 2.$$

$$a_n = \frac{1}{2} \times 2 \int_0^2 x \cos(\frac{n\pi x}{2}) =$$
$$[(x)(\frac{2}{n\pi}\sin(\frac{n\pi x}{2}) - (1)(((\frac{2}{n\pi})^2(-\cos(\frac{n\pi x}{2})))]_0^2$$
$$(\frac{2}{n\pi})^2((-1)^n - 1).$$

So on $[-2, 2]$

$$|x| = 1 + \sum_{n=1}^{\infty} (\frac{2}{n\pi})^2((-1)^n - 1)\cos(\frac{n\pi x}{2}).$$

(b) Compute the solution $u(t, x)$ for the partial differential equation on the interval $[0, 2]$:

$$9u_t = u_{xx} \quad \text{with}$$
$$u(t,0) = u(t,2) = 0 \quad \text{for } t > 0 \qquad \text{(boundary conditions)}$$
$$u(0,x) = 8\sin(10\pi x) - 10\sin(8\pi x) \quad \text{for } 0 < x < 2 \qquad \text{(initial conditions)}$$

Since $u_t = \frac{1}{9}u_{xx}$, $\quad \alpha = \frac{1}{3}$

$$u(t,x) = 8exp[-(\frac{10\pi}{3})^2 t]\sin(10\pi x) - 10exp[-(\frac{8\pi}{3})^2 t]\sin(8\pi x).$$