(1)(20 points) (a) For the differential equation $y'' + 9y' = 0$ obtain a fundamental pair of solutions and compute the Wronskian of the pair.

(b) Solve the initial value problem: $y'' + 9y' = 12e^{3t} + 9$ with $y(0) = 0$ and $y'(0) = 0$.

(a) The characteristic polynomial for the homogeneous equation is $r^2 + 9r = r(r+9)$ with roots $0, -9$. The fundamental pair is $\{1 = e^{0t}, e^{-9t}\}$ and so the Wronskian $= y_1 y_2' - y_2 y_1' = 1 \cdot (-9e^{-9t}) - (e^{-9t}) \cdot 0 = -9e^{-9t}$.

(b) The general solution of the homogeneous system is $y_h = C_1 + C_2 e^{-9t}$.

The associated root for $12e^{3t}$ is 3 and the associated root for 9 is 0. So the first version of the test function is $Y^1 = Ae^{3t} + B$. Because 0 is a root of the homogeneous equation, the test function is $Y = Ae^{3t} + Bt$.

$$\begin{array}{lrcl} 9\times & Y' & = & 27Ae^{3t} + 9B \\ 1\times & Y_p'' & = & 9Ae^{3t} \end{array}$$

$$12e^{3t} + 9 = 0 + 36Ae^{3t} + 9B$$

So $A = 1/3,\ B = 1$.

$$\begin{aligned} y_g &= C_1 + C_2 e^{-9t} + \frac{1}{3}e^{3t} + t \\ y_g' &= +(-9C_2)e^{-9t} + e^{3t} + 1. \end{aligned}$$

Since $y_g(0) = y_g'(0) = 0.$, $0 = C_1 + C_2 + \frac{1}{3}$ and $0 = -9C_2 + 2$. So $C_2 = 2/9, C_1 = -5/9$.

$y_g = (-5/9) + (2/9)e^{-9t} + (1/3)e^{3t} + t$.

(2) (16 points) Compute the general solution of the homogeneous equation $y^{(6)} + 2y^{(3)} + y = 0$.

The characteristic equation $0 = r^6 + 2r^3 + 1 = (r^3+1)^2$. As the difference of two cubes $r^3 + 1 = (r+1)(r^2 - r + 1)$ with roots $-1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$. Alternatively, use DeMoivre's Theorem using three steps of $120°$ each around the circle of radius 1, beginning with a half-step of$60°$. As each root is repeated twice the solution is

$$C_1 e^{-t} + C_2 te^{-t} + C_3 e^{t/2}\cos(\sqrt{3}t/2) +$$
$$C_4 te^{t/2}\cos(\sqrt{3}t/2) + C_5 e^{t/2}\sin(\sqrt{3}t/2) + C_6 te^{t/2}\sin\sqrt{3}t/2).$$

(3)(20 points) Consider the twelfth order differential equation $y^{(12)} - 2y^{(8)} + y^{(4)} = g(t)$.

(a) Compute the general solution of the homogeneous equation with $g(t) = 0$.

(b) For each of the following forcing functions $g(t)$, write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients. Do not solve for the constants.

$$(i) \quad g(t) = t^2 + te^t + 2. \qquad (ii) \quad g(t) = 2e^t \sin(t) + te^{-t}.$$

(a) The characteristic polynomial is $r^{12} - 2r^8 + r^4 = r^4(r^4 - 1)^2 = r^4(r^2 - 1)^2(r^2 + 1)^2$. The roots are $0, 0, 0, 0, 1, 1, -1, -1, \pm\mathbf{i}, \pm\mathbf{i}$.

$$y_h = C_1 + C_2 t + C_3 t^2 + C_4 t^3 + C_5 e^t + C_6 te^t +$$
$$C_7 e^{-t} + C_8 te^{-t} + C_9 \cos(t) + C_{10} \sin(t) + C_{11} t\cos(t) + C_{12} t\sin(t).$$

(b) (i) Associated root for t^2+2 is 0 and for t^2e^t is 1. So $Y_p^1 = [At^2+Bt+C] + [(Dt+E)e^t]$ and as 0 is a root of the homogeneous repeated four times and 1 is a root of the homogeneous repeated twice $Y_p^1 = t^4[At^2+Bt+C] + t^2[(Dt+E)e^t]$.

(ii) Associated roots for $2e^t\sin(t)$ are $1 \pm \mathbf{i}$, and for te^{-t} is -1

So $Y_p^1 = [Ae^t\cos(t) + Be^t\sin(t)] \ + \ [(Ct+D)e^{-t}]$.

None of the associated roots are roots of the homogeneous and so $Y_p = Y_p^1$.

(4)(16 points) Compute the general solution of $y'' + 2y' + y = t^{-1}e^{-t}$.

The characteristic polynomial for the homogeneous equation is $r^2 + 2r + 1 = (r+1)(r+1)$ with roots $-1, -1$. The solution of the homogeneous is $y_h = C_1 e^{-t} + C_2 te^{-t}$.

We look for a particular solution $y_p = u_1 e^{-t} + u_2 te^{-t}$.

The equations are:

$$\begin{aligned} u_1' e^{-t} + u_2' te^{-t} &= 0, \\ u_1'(-e^{-t}) + u_2'(e^{-t} - te^{-t}) &= t^{-1}e^{-t} \end{aligned}$$

The Wronskian is e^{-2t}.

$u_1' = -e^{-2t}/e^{-2t} = -1, \quad u_2' = t^{-1}e^{-2t}/e^{-2t} = t^{-1}$.

So $u_1 = -t, u_2 = \ln t$ and $y_p = -te^{-t} + te^{-t}\ln t$.

$$y_g = C_1 e^{-t} + C_2 te^{-t} - te^{-t} + te^{-t}\ln t \ \text{ or } \ C_1 e^{-t} + C_2 te^{-t} + te^{-t}\ln t.$$

(5)(16 points) For the differential equation, $(x-1)y''-xy'+y=0$, $y_1=e^x$ is a solution. Use the method of Reduction of Order to compute a second solution y_2 which is independent of the first one.

Look for $y_2 = ue^x$ So that

$$\begin{array}{rrcl}
1\times & y_2 & = & ue^x \\
-x\times & y_2' & = & ue^x \;+\; ue^x \\
(x-1)\times & y_2'' & = & ue^x \;+\; 2u'e^x \;+\; u''e^x \\
\\
 & 0 & = & 0 \quad +\; (x-2)u'e^x \;+\; (x-1)u''e^x.
\end{array}$$

With $v=u'$, $\frac{dv}{v} = -\frac{(x-2)dx}{(x-1)} = (-1+\frac{1}{x-1})dx$

Therefore, $\ln v = -x + \ln(x-1)$ and so $\frac{du}{dx} = (x-1)e^{-x}$ and $u = -xe^{-x}$

$$y_2 = ue^x = -x.$$

(6)(12 points) A hanging spring is stretched 6 inches (= .5 feet) by a weight of 64 pounds.

(a) Set up the initial value problem (differential equation and initial conditions) which describes the motion, neglecting friction, when the weight is pulled down an additional foot and is then released and is subjected to an external force of $6\cos(\omega t)$. You need not solve the equation. (Recall that g, the acceleration due to gravity is 32 feet/second^2.)

(b) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when $\omega = 4$.

(c) Write down the test function with the fewest terms which can be used to obtain a particular solution via the Method of Undetermined Coefficients when ω is the *resonance frequency* that is, the natural frequency of the spring.

(a) $mg = w = 64, \quad m = 2\ \Delta L = 1/2$ and so $w = k\Delta L$ implies $k = 128$.

The equation is

$$2y'' + 0y' + 128y = 6\cos(\omega t) \quad \text{or} \quad y'' + 64y = 3\cos(\omega t)$$

with initial conditions $y(0) = -1, y'(0) = 0$.

The characteristic polynomial for the homogeneous equation is $r^2 + 64$ with roots $\pm 8\mathbf{i}$. Thus, the natural frequency of the spring is 8.

(b) $Y_p = A\cos(4t) + B\sin(4t)$.

(c) $Y_p = t[A\cos(8t) + B\sin(8t)]$.

Good luck. Remember to show your work.