(1) (12 points each)Compute the general solution of each of the following differential equations:

(a) $\frac{dy}{dx} \quad = \quad \frac{x}{y+xy}.$

Variables Separable: $\int y\frac{dy}{} = \int \frac{x}{1+x}dx$.
Let $u = 1 + x$ so that $x = u - 1$.
$\int \frac{u-1}{u}du \ = \ u - \ln(u) + C \ = \ x - \ln(x+1) + C$.
So $y^2/2 \ = \ x - \ln(x+1) + C$.

(b) $x\frac{dy}{dx} + y \quad = \quad xy + e^x$

Linear: $x\frac{dy}{dx} + (1-x)y \quad = \quad e^x$
Divide by x. $\frac{dy}{dx} + \frac{1-x}{x}y \quad = \quad \frac{e^x}{x}$
$\mu = exp(\int \frac{1-x}{x}dt) = exp(\ln x - x) = xe^{-x}$
$xe^{-x}\frac{dy}{dx} \ + (1-x)e^{-x}y \quad = \quad [xe^{-x}y]' \quad = \quad 1.$
So $xe^{-x}y \ = \ x + C$ or $y \ = \ e^x + C\frac{e^x}{x}$.

(c) $y'' - 3y \ = \ 0$ (y is a function of x).

Second Order Linear: Characteristic Equation $r^2 - 3 = 0$ with roots $r = \pm\sqrt{3}$.
$y \quad = \quad C_1 \ e^{-\sqrt{3}x} + \ C_2e^{-\sqrt{3}x}$.

(d) $y \ \frac{dy}{dx} \quad = \quad (5y - 6x)$.

Homogeneous with $\frac{dy}{dx} = \frac{5y-6x}{y}$. Let $z = y/x$.
$x\frac{dz}{dx} = \frac{5z-6}{z} - z = -\frac{z^2-5z+6}{z}$.
$-\int \frac{z}{(z-3)(z-2)}dz = \int -\frac{dx}{x}$.
$\frac{z}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z-2}$. So $z = A(z-2) + B(z-3)$.
With $z = 3, A = 3$ and with $z = 2, B = -2$.

$$C - \ln(x) = 3\ln(\frac{y-3x}{x}) - 2\ln(\frac{y-2x}{x}).$$

$$3\ln(y-3x) + \ln(y-2x) = C, \quad \text{or} \quad (y-3x)(y-2x) = C.$$

(2) (12 points each) Solve the following initial value problems:

(a) $\quad y'' + t(y')^2 = 0, \qquad$ with $\quad y(0) = 2, y'(0) = 1.$

Let $y' = v$ so that $y'' = \frac{dv}{dt}$.
$\frac{dv}{dt} = -tv^2$. $-v^{-1} = \int v^{-2}dv = \int -tdt = -t^2/2 + C_1$.
When $t = 0$, $v = 1$ and so $C_1 = -1$.
$\frac{dy}{dt} = v = \frac{2}{2+t^2} = \frac{1}{1+(t/\sqrt{2})^2}$. So $y = \sqrt{2}\arctan(t/\sqrt{2}) + C_2$.
When $t = 0$, $y = 2$ and so $C_2 = 2$. $y = \sqrt{2}\arctan(t/\sqrt{2}) + 2$.

(b) $\quad (1+e^{xy}(y\cos x - \sin x))\,dx + (xe^{xy}\cos x - y)\,dy \quad = \quad 0$ with $\quad y(0) = 2\pi$.

$\frac{\partial}{\partial y}(1+e^{xy}(y\cos x - \sin x)) = e^{xy}(xy\cos x - x\sin x + \cos x) = \frac{\partial}{\partial x}(xe^{xy}\cos x - y)$
Exact Equation: $F = \int (xe^{xy}\cos x - y)\,dy = e^{xy}\cos x - \frac{y^2}{2} + H(x)$.
$\frac{\partial F}{\partial x} = e^{xy}(y\cos x - \sin x) + H'(x) = 1 + e^{xy}(y\cos x - \sin x)$.
$H'(x) = 1$ and so $H(x) = x$.
$e^{xy}\cos x - \frac{y^2}{2} + x = C$. Since $y(0) = 2\pi$, $1 - 2\pi^2 + 0 = C$ and so $C = 1 - 2\pi^2$.

(3)(9 points) (a) Assume that y_1, y_2 are solutions of the equation $y'' + py' + qy = 1$ where p and q are functions of t. Show that $y_1 + y_2$ is NOT a solution of this equation.

Because y_1 and y_2 are solutions we have
$y_1'' + py_1' + qy_1 = 1$ and $y_2'' + py_2' + qy_2 = 1$. Adding, we get that

$$(y_1 + y_2)'' + p(y_1 + y_2)' + q(y_1 + y_2) = 2 \neq 1.$$

(b) Compute the *Wronskian* W of the pair $y_1 = e^{3x}, y_2 = xe^x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^x \\ 3e^{3x} & (1+x)e^x \end{vmatrix}$$

$$= (1+x)e^{3x} - 3xe^{3x} = (1-2x)e^{3x}.$$

So $W = 0$ when $x = \frac{1}{2}$. Notice that this does not violate Abel's Theorem as these two functions are not solutions of a second order linear, homogeneous equation defined around $x = \frac{1}{2}$.

(4) (10 points)A 500 gallon tank contains 150 gallons of water in which is dissolved 25 pounds of salt. Starting at time t = 0, a solution with a concentration of 3 pounds per gallon is pumped into the tank at a rate of 5 gallons per minute. At the same time, the well-stirred mixture is pumped out at the rate of 2 gallons per minute.

Set up an initial value problem (differential equation and initial conditions) for the amount $Q(t)$ of salt (in pounds) in the tank at time t until the tank is full. You need not solve the equation.

$\frac{dV}{dt} = 5 - 2$ in gal/min with $V_0 = 150$. So $V = 150 + 3t$.
In $pounds/min$ $\frac{dQ}{dt} = 5 \cdot 3 - 2\frac{Q}{V}$.
$\frac{dQ}{dt} = 15 - \frac{2}{150+3t}Q$ with $Q_0 = 25$.

(5) (9 points) I borrow \$20,000 at an interest rate of 3% per year, compounded continuously. I pay off the loan continuously at a rate of \$1100 per year. Set up an initial value problem (differential equation and initial conditions) whose solution is the quantity $S(t)$ of dollars that I owe at time t, until the loan is paid off. You need not solve the equation.

$\frac{dP}{dt} = .03P - 1100$, with $P_0 = 20,000$.

Remember to show your work. Good luck.