Math 39100 K (32336) - Homework Solutions 03

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Contents

Example 5.2/ BD9 : $(1+x^2)y'' - 4xy' + 6y = 0$

$$1y'' = \Sigma\ 1n(n-1)a_n x^{n-2} [k = n-2] = \Sigma (k+2)(k+1)a_{k+2}x^k.$$
$$+x^2y'' = \Sigma\ n(n-1)a_n x^n\ [k=n] = \Sigma\ k(k-1)a_k x^k.$$
$$-4xy' \quad = \quad \Sigma\ -4na_n x^n \quad [k=n] = \quad \Sigma\ -4ka_k x^k.$$
$$6y \quad = \quad \Sigma\ 6a_n x^n \qquad [k=n] = \quad \Sigma\ 6a_k x^k.$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[(-k(k-1)+4k-6)a_k] = \frac{-k^2+5k-6}{(k+2)(k+1)}a_k$$

$$= -\frac{(k-2)(k-3)}{(k+2)(k+1)}a_k.$$

$k = 0 : a_2 = -3a_0,$ $\qquad k = 1 : a_3 = -\frac{1}{3}a_1.$
$k = 2 : a_4 = 0,$ $\qquad k = 3 : a_5 = 0.$ So $a_k = 0$ for $k \geq 2$.

Example 5.2/ BD12,18; BDM14 :
$(1-x)y'' + xy' + -y = 0, y(0) = -3, y'(0) = 2.$

$$1y'' = \Sigma\ 1n(n-1)a_n x^{n-2}[k=n-2] = \Sigma(k+2)(k+1)a_{k+2}x^k$$

$$-xy'' = \Sigma\ -n(n-1)a_n x^{n-1}\ [k=n-1] = \Sigma\ -(k+1)(k)a_{k+1}x^k.$$

$$+xy' = \Sigma\ na_n x^n\ [k=n] = \Sigma\ ka_k x^k.$$

$$-y = \Sigma\ -a_n x^n\ [k=n] = \Sigma\ -a_k x^k.$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[(k+1)ka_{k+1} - (k-1)a_k], \quad a_0 = -3, a_1 = 2.$$

$k=0: a_2 = a_0/2 = -3/2,$
$k=1: a_3 = a_2/3 = a_0/6 = -1/2.$
$k=2: a_4 = [6a_3 - a_2]/12 = -a_0/24 = 1/8,$
$k=3: a_5 = [12a_4 - 2a_3]/20 = -a_0/24 = 1/8$
Where is a_1 ?

Example 5.2/ BD8; BDM8 : $xy'' + y' + xy = 0, \quad x_0 = 1.$
Let $X = x - 1$ and so $x = X + 1$ so that the equation becomes $(1 + X)y'' + y' + (1 + X)y = 0.$

$$1y'' = \Sigma\, n(n-1)a_n X^{n-2} [k = n-2] = \Sigma (k+2)(k+1)a_{k+2}X^k.$$
$$+Xy'' = \Sigma\, n(n-1)a_n X^{n-1}\ [k = n-1] = \Sigma\ (k+1)(k)a_{k+1}X^k.$$
$$+y' = \Sigma\, na_n X^{n-1}\ [k = n-1] = \Sigma\ (k+1)a_{k+1}X^k.$$
$$+y = \Sigma\, a_n X^n\ [k = n] = \Sigma\ a_k X^k.$$
$$+Xy = \Sigma\, a_n X^{n+1}\ [k = n+1] = \Sigma\ a_{k-1}X^k.$$

$$a_{k+2} = -\frac{1}{(k+2)(k+1)}[(k+1)ka_{k+1} + (k+1)a_{k+1} + a_k + a_{k-1}].$$

$$a_{k+2} = -\frac{1}{(k+2)(k+1)}[(k+1)^2 a_{k+1} + a_k + a_{k-1}].$$

$$k=0, \quad a_2 = -\frac{1}{2}[a_1 + a_0 + 0] = -\frac{1}{2}[a_1 + a_0].$$

$$k=1, \quad a_3 = -\frac{1}{6}[4a_2 + a_1 + a_0] = \frac{1}{6}[a_1 + a_0].$$

$$k=2, \quad a_4 = -\frac{1}{12}[9a_3 + a_2 + a_1] = -\frac{1}{12}[2a_1 + a_0].$$

$$y = a_0 + a_1(x-1) - \frac{1}{2}[a_1 + a_0](x-1)^2 +$$

$$\frac{1}{6}[a_1 + a_0](x-1)^3 - \frac{1}{12}[2a_1 + a_0](x-1)^4 + \dots$$

Example 5.3/ BD3 :
$x^2y'' + (1+x)y' + 3(\ln x)y = 0, \quad y(1) = 2, y'(1) = 0.$

At $x = 1, \quad y''(1) + 2 \cdot 0 + 3 \cdot 0 \cdot 2 = 0$, and so $y''(1) = 0$.

$$x^2y^{(3)} + 2xy'' + (1+x)y'' + y' + 3(\ln x)y' + (3/x)y = 0.$$

At $x = 1, \quad y^{(3)}(1) + 0 + 0 + 0 + 3 \cdot 2 = 0,$

and so $y^{(3)}(1) = -6$.

$$x^2y^{(4)} + 2xy^{(3)} + 2xy^{(3)} + 2y'' + (1+x)y^{(3)} + y'' + y''$$
$$+3(\ln x)y'' + (3/x)y' + (3/x)y' + (-3/x^2)y = 0.$$

$$x^2y^{(4)} + 2xy^{(3)} + 2xy^{(3)} + 2y'' + (1+x)y^{(3)} + y'' + y''$$
$$+3(\ln x)y'' + (3/x)y' + (3/x)y' + (-3/x^2)y = 0.$$

At $x = 1, \quad y^{(4)}(1) + 2\cdot(-6) + 2\cdot(-6) + 0 + 2\cdot(-6) + 0 + 0$
$+0 + 0 + 0 + 0 + (-3)\cdot 2 = 0.$

So $y^{(4)}(1) = 42.$

Example 5.5/ BD5; : $3x^2y'' + 2xy' + x^2y = 0$ We will look for solutions with $a_0 = 1$.
For the associated Euler equation write:
$x^2[3]y'' + x[2]y' + [x^2]y = 0$. So the associated Euler equation is $3x^2y'' + 2xy' + 0 = 0$ with indicial equation

$$0 = 3r(r-1) + 2r = r(3r-1), \quad \text{with roots } \ 0, 1/3.$$

$y = x^r \Sigma a_n x^n = \Sigma a_n x^{n+r}$.

$$3x^2y'' = \Sigma\ 3(n+r)(n+r-1)a_n x^{n+r}[k=n]$$
$$= \Sigma 3(k+r)(k+r-1)a_k x^{k+r}$$
$$+2xy' = \Sigma\ 2(n+r)a_n x^{n+r} \quad [k=n] = \quad \Sigma\ 2(k+r)a_k x^{k+r}.$$
$$+x^2y = \Sigma\ a_n x^{n+r+2}[k=n+2] = \quad \Sigma\ a_{k-2} x^{k+r}.$$

$$[3(k+r)(k+r-1)+2(k+r)]a_k \ = \ (k+r)(3(k+r)-1)a_k \ = \ -a_{k-2}.$$

$$(k+r)(3(k+r)-1)a_k \;=\; -a_{k-2}.$$

$r = 1/3 : a_k = \frac{-1}{k(3k+1)} a_{k-2}$. So $a_k = 0$ if k is odd. $a_0 = 1$:

$$k = 2 : a_2 = \frac{-1}{14}; \qquad k = 4 : a_4 = \frac{-1}{52} a_2 = \frac{1}{728}$$

$$y_1 = x^{1/3}[1 + \frac{-x^2}{14} + \frac{x^4}{728} + \dots$$

$r = 0 : a_k = \frac{-1}{k(3k-1)} a_{k-2}$. So $a_k = 0$ if k is odd. $a_0 = 1$:

$$k = 2 : a_2 = \frac{-1}{10}; \qquad k = 4 : a_4 = \frac{-1}{44} a_2 = \frac{1}{440}$$

$$y_1 = x^0[1 + \frac{-x^2}{10} + \frac{x^4}{440} + \dots$$

Example 5.5/ BD6; BDM5 ; : $x^2y'' + xy' + (x-2)y = 0$ We will look for solutions with $a_0 = 1$.
For the associated Euler equation write:
$x^2[1]y'' + x[1]y' + [x-2]y = 0$. So the associated Euler equation is $x^2y'' + xy' - 2y = 0$ with indicial equation

$$0 = r(r-1) + r - 2 = (r+\sqrt{2})(r-\sqrt{2}), \quad \text{with roots} \pm\sqrt{2}.$$

$$y = x^r \Sigma a_n x^n = \Sigma a_n x^{n+r}.$$

$$x^2y'' = \Sigma\ (n+r)(n+r-1)a_n x^{n+r}[k=n] = \Sigma(k+r)(k+r-1)a_k$$

$$+xy' = \Sigma\ (n+r)a_n x^{n+r} \quad [k=n] = \quad \Sigma\ (k+r)a_k x^{k+r}.$$

$$-2y = \Sigma\ -2a_n x^{n+r}[k=n] = \quad \Sigma\ -2a_k x^{k+r}.$$

$$+xy = \Sigma\ a_n x^{n+r+1}[k=n+1] = \quad \Sigma\ a_{k-1} x^{k+r}.$$

$$[(k+r)(k+r-1)+(k+r)-2]a_k \ = \ (k+r+\sqrt{2})(k+r-\sqrt{2})a_k$$

$$= \qquad -a_{k-1}.$$

$$(k + r + \sqrt{2})(k + r - \sqrt{2})a_k \ = \ -a_{k-1}.$$

With $r = \sqrt{2}$, $\quad a_k = -\frac{1}{k(k+2\sqrt{2})}a_{k-1}.$

$$a_1 = -\frac{1}{1 + 2\sqrt{2}}; \qquad a_2 = \frac{1}{2(1 + 2\sqrt{2})(2 + 2\sqrt{2})}$$

$$a_3 = -\frac{1}{6(1 + 2\sqrt{2})(2 + 2\sqrt{2})(3 + 2\sqrt{2})}$$

$$y \ = \ x^{\sqrt{2}}[1 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Example 5.5/ BD7; BDM6 : $xy'' + (1-x)y' - y = 0$.
Multiply by x: $x^2y'' + (x - x^2)y' - xy = 0$.
For the associated Euler equation write:
$x^2[1]y'' + x[1-x]y' + [-x]y = 0$. So the associated Euler equation is $x^2y'' + xy' + 0 = 0$ with indicial equation

$$0 = r(r-1) + r = r^2, \quad \text{with roots } \ 0, 0.$$

$y = x^r \Sigma a_n x^n = \Sigma a_n x^{n+r}$.

$$x^2y'' = \Sigma\ (n+r)(n+r-1)a_n x^{n+r}[k=n]$$
$$= \Sigma(k+r)(k+r-1)a_k x^{k+r}$$
$$+xy' = \Sigma\ (n+r)a_n x^{n+r} \quad [k=n] = \Sigma\ (k+r)a_k x^{k+r}.$$
$$-x^2y' = \Sigma\ -(n+r)a_n x^{n+r+1}[k=n+1]$$
$$= \Sigma\ -(k+r-1)a_{k-1}x^{k+r}.$$
$$-xy = \Sigma\ -a_n x^{n+r+1}[k=n+1] = \quad \Sigma\ -a_{k-1}x^{k+r}.$$

$$[(k+r)(k+r-1)+(k+r)]a_k \;=\; (k+r)^2 a_k \;=\; (k+r)a_{k-1}.$$

We obtain a solution for only one root, $r=0$.
$r=0: a_k = \frac{1}{k}a_{k-1}$. So $k=1: a_1=1$

$$k=2: a_2=\frac{1}{2}a_1=\frac{1}{2!}; \qquad k=3: a_3=\frac{1}{3}a_2=\frac{1}{3!}$$

$$y_1 = x^0[1+x+\frac{x^2}{2!}+\frac{x3}{3!}+\dots$$

This pattern should look familiar, in fact, it is $y_1=e^x$ although you are not required to recognize this.
But you should remember that, in theory at least, if you have one solution how you get another one.
No. No. Don't just multiply by x, but you are getting warm.

Every once in awhile an equation can be solved by some trick. In this case if you go back to the original equation $xy'' + (1-x)y' - y = 0$. Notice that

$$0 = xy''+(1-x)y'-y = xy''+y'-xy'-y = (xy')'-(xy)' = (xy'-xy)'$$

So $C_1 = xy' - xy$, which is a linear equation $y' - y = C_1 x^{-1}$ with integrating factor $\mu = e^{-x}$
Hence, $y = C_2 e^x + C_1 e^x \cdot \int x^{-1} e^{-x} dx$.
In particular, e^x is a solution.

Example 5.5/ BD9:

$x^2y''-x(x+3)y'+(x+3)y = x^2y''+(-x^2-3x)y'+(x+3)y = 0$

We will look for solutions with $a_0 = 1$.

For the associated Euler equation write:

$x^2[1]y'' + x[-x-3]y' + [x+3]y = 0$. So the associated Euler equation is $x^2y'' - 3xy' + 3 = 0$ with indicial equation $0 = r(r-1) - 3r + 3 = r^2 - 4r + 3 = (r-1)(r-3)$with roots $1, 3$.

$$x^2y'' = \Sigma\ (n+r)(n+r-1)a_nx^{n+r}[k=n]$$
$$= \Sigma(k+r)(k+r-1)a_kx^{k+r}$$
$$-x^2y' = \Sigma\ -(n+r)a_nx^{n+r+1}[k=n+1]$$
$$= \Sigma\ -(k+r-1)a_{k-1}x^{k+r}.$$
$$-3xy' = \Sigma\ -3(n+r)a_nx^{n+r}[k=n] = \Sigma\ -3(k+r)a_kx^{k+r}.$$
$$+xy = \Sigma\ a_nx^{n+r+1}[k=n+1] = \Sigma\ a_{k-1}x^{k+r}.$$
$$+3y = \Sigma\ 3a_nx^{n+r}\quad [k=n] = \Sigma\ 3a_kx^{k+r}.$$

$[(k+r)(k+r-1)-3(k+r)+3]a_k \;=\; (k+r-1)(k+r-3)a_k$
$= \;(k+r-2)a_{k-1}.$
We can only use the larger root $r = 3$.

$$a_k = \frac{k+1}{k(k+2)}a_{k-1}.$$

$k = 1 : a_1 = \frac{2}{3}a_0 = \frac{2}{3}. \quad k = 2 : \frac{3}{8}a_1 = \frac{1}{4}.$

$$y_1 = x^3[1 + \frac{2}{3}x + \frac{1}{4}x^2 + \cdots$$

Example 6.2/ BD22; BDM16 :
$y'' - 2y' + 2y = e^{-t}, \quad y(0) = 0, y'(0) = 1$

$$\mathcal{L}(y'') - 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(e^{-t})$$

$$[s^2\mathcal{L}(y) - y'(0) - sy(0)] - 2[s\mathcal{L}(y) - y(0)] + 2[\mathcal{L}(y)]$$
$$= [s^2 - 2s + 2]\mathcal{L}(y) - 1 = \frac{1}{s+1}.$$

So $\mathcal{L}(y) = \frac{s+2}{(s+1)(s^2-2s+2)}$.

Example 10.2/ BD13; BDM13 :
$f(x) = -x, \quad -L \leq x < L; \quad f(x+2L) = f(x).$

Odd function, so $a_n = 0$ for all n.

$b_n = \frac{2}{L}\int_0^L (-x)\sin(\frac{n\pi}{L}x)\ dx\ =$
$\frac{2}{L}[(-x)(-\frac{L}{n\pi}\cos(\frac{n\pi}{L}x)) \quad - \quad (-1)(-(\frac{L}{n\pi})^2\sin(\frac{n\pi}{L}x))]_0^L.$

$\sin(n\pi) = 0, \cos(n\pi) = (-1)^n$. Therefore,

$b_n \ = \ \frac{2}{L}[(-L)(-\frac{L}{n\pi}(-1)^n) \ = \ \frac{2L(-1)^n}{n\pi}$

$f(x) = \Sigma_{n=1}^{\infty} \ \frac{2L(-1)^n}{n\pi}\sin(\frac{n\pi}{L}x).$

Example 10.2/ BD16; BDM 16 :

$$f(x) = \begin{cases} x+1, & -1 \le x < 0, \\ 1-x, & 0 \le x < 1; \end{cases} \quad f(x+2) = f(x).$$

Even function, so $b_n = 0$ for all n. Here $L = 1$.

$a_0 \ = \ 2\int_0^1 f(x)dx = 2\int_0^1 1 - xdx = 2[x - x^2/2]_0^1 = 1$. For $n \geq 1$

$a_n = 2\int_0^1 (1-x)\cos(n\pi x)dx =$
$2[(1-x)(\frac{1}{n\pi}\sin(n\pi x)) - (-1)(-(\frac{1}{n\pi})^2\cos(n\pi x)]_0^1.$

$a_n = -2(\frac{1}{n\pi})^2[(-1)^n - 1] = 4(\frac{1}{n\pi})^2$ if n is odd and $= 0$ if n is even.

$f(x) \ = \ \frac{1}{2} \ + \ \Sigma_{n,odd=1}^{\infty} 4(\frac{1}{n\pi})^2\cos(n\pi x)$
$= \frac{1}{2} \ + \ \Sigma_{k=1}^{\infty} 4(\frac{1}{(2k-1)\pi})^2\cos((2k-1)\pi x).$

Example 10.2/ BD17; BDM17 :

$$f(x) = \begin{cases} x + L, \ -L \leq x \leq 0, \\ L, \ 0 < x < L; \end{cases} \quad f(x + 2L) = f(x).$$

$$a_0 = \frac{1}{L}[\int_{-L}^{0} x + Ldx + \int_{0}^{L} Ldx] = \frac{1}{L}[-\frac{L^2}{2} + 2L^2] = \frac{3L}{2}.$$

$a_n = \frac{1}{L}[\int_{-L}^{0}(x+L)\cos(\frac{n\pi}{L}x)dx + \int_0^L L\cos(\frac{n\pi}{L}x)dx]$
$= \frac{1}{L}[\int_{-L}^{0} x\cos(\frac{n\pi}{L}x)dx + 2\int_0^L L\cos(\frac{n\pi}{L}x)dx] =$
$\frac{1}{L}[[(x)(\frac{L}{n\pi}\sin(\frac{n\pi}{L}x) - (1)(-(\frac{L}{n\pi})^2\cos(\frac{n\pi}{L}x))]_{-L}^{0} +$
$[2L(\frac{L}{n\pi}\sin(\frac{n\pi}{L}x)]_0^L.$

$a_n = \frac{1}{L}[(\frac{L}{n\pi})^2[1-(-1)^n] = \frac{2L}{(n\pi)^2}$ if n is odd and $= 0$ if n is even.

$$b_n = \frac{1}{L}[\int_{-L}^{0}(x+L)\sin(\frac{n\pi}{L}x)dx + \int_0^L L\sin(\frac{n\pi}{L}x)dx]$$

$$= \frac{1}{L}[\int_{-L}^{0} x\sin(\frac{n\pi}{L}x)dx + 0] =$$

$$\frac{1}{L}[(x)(-\frac{L}{n\pi}\cos(\frac{n\pi}{L}x)) - (1)(-(\frac{L}{n\pi})^2\sin(\frac{n\pi}{L}x))]_{-L}^{0}$$

$$b_n = \frac{1}{L}[(L)\frac{L}{n\pi}(-1)^n]$$

$$f(x) = \frac{3L}{4} + \Sigma_{n,odd=1}^{\infty} \frac{2L}{(n\pi)^2} \cos(\frac{n\pi}{L} x)$$

$$+\Sigma_{n=1}^{\infty} (-1)^n \frac{L}{n\pi} \sin(\frac{n\pi}{L} x).$$

Example 10.2/ BD18; BDM18 :

$$f(x) = \begin{cases} 0, & -2 \le x \le -1, \\ x, & -1 < x < 1, \\ 0, & 1 \le x < 2 \end{cases} \qquad f(x+4) = f(x).$$

So $L = 2$.

Odd function, so $a_n = 0$ for all n.

$$b_n = \frac{2}{2}[\int_0^2 f(x)\sin(\frac{n\pi}{2}x)dx] = \int_0^1 x\sin(\frac{n\pi}{2}x)dx$$

$$= [(x)(-\frac{2}{n\pi}\cos(\frac{n\pi}{2}x)) - (1)(-(\frac{2}{n\pi})^2\sin(\frac{n\pi}{2}x))]_0^1$$

$$-\frac{2}{n\pi}\cos(\frac{n\pi}{2}) + (\frac{2}{n\pi})^2\sin(\frac{n\pi}{2}).$$

Notice that $\cos(\frac{n\pi}{2}) = 0$ if n is odd, but is $(-1)^{n/2}$ if n is even. That is, if $n = 2k$ then $\cos(\frac{n\pi}{2}) = (-1)^k$.

This time the sine terms are not all zero. $\sin(\frac{n\pi}{2}) = 0$ if n is even, but if $n = 2k-1$ for $k = 1, 2...$ then $\sin(\frac{n\pi}{2}) = (-1)^{k+1}$. That is, $\sin(\frac{\pi}{2}) = 1, \sin(\frac{3\pi}{2}) = -1, ...$

So we can write

$$b_n = \begin{cases} (-1)^{k+1}\frac{1}{k\pi} & n = 2k, \\ (-1)^{k+1}(\frac{2}{(2k-1)\pi})^2 & n = 2k-1. \end{cases}$$

However, this kind of analysis you won't have to do. You will have to notice that the sine terms are not all zero, but you can leave the answer as:

$$b_n = -\frac{2}{n\pi}\cos(\frac{n\pi}{2}) + (\frac{2}{n\pi})^2\sin(\frac{n\pi}{2}).$$

So that
$f(x) = \Sigma_{n=1}^{\infty} \; [-\frac{2}{n\pi}\cos(\frac{n\pi}{2}) + (\frac{2}{n\pi})^2\sin(\frac{n\pi}{2})]\sin(\frac{n\pi}{2}x).$

Example 10.4/ BD20; BDM20 : $f(x) = x, \ 0 \leq x < 1$. Cosine series with $L = 1$.

$$a_0 = \frac{2}{1}[\int_0^1 f(x)dx] = 2\int_0^1 xdx = x^2|_0^1 = 1.$$

For $n > 0$

$$a_n = \frac{2}{1}[\int_0^1 f(x)\cos(n\pi x)dx] = 2[\int_0^1 x\cos(n\pi x)dx] =$$

$$2[(x)(\frac{1}{n\pi}\sin(n\pi x)) - (1)(-(\frac{1}{n\pi})^2\cos(n\pi x)]_0^1$$

$$= \frac{2((-1)^n - 1)}{n^2\pi^2}.$$

$$x = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{n^2\pi^2}\cos(n\pi x).$$

Example 10.4/ BD16; BDM16 : $f(x) = \begin{cases} x, & 0 \le x < 1, \\ 1, & 1 \le x < 2; \end{cases}$.

Sine series with $L = 2$.

$$b_n = \frac{2}{2}[\int_0^2 f(x)\sin(\frac{n\pi}{2}x)dx] = \int_0^1 x\sin(\frac{n\pi}{2}x)dx + \int_1^2 \sin(\frac{n\pi}{2}x)dx$$

The first integral, from 10.2/18 is $-\frac{2}{n\pi}\cos(\frac{n\pi}{2}) + (\frac{2}{n\pi})^2\sin(\frac{n\pi}{2})$.

The second integral is

$$-\frac{2}{n\pi}\cos(\frac{n\pi}{2}x)|_1^2 = (-1)^{n+1}\frac{2}{n\pi} + \frac{2}{n\pi}\cos(\frac{n\pi}{2}).$$

Putting them together, the first and last terms cancel, so that

$$b_n = (\frac{2}{n\pi})^2 \sin(\frac{n\pi}{2}) + (-1)^{n+1}\frac{2}{n\pi}.$$

Suppose we wish to solve the heat equation
$81u_{xx} = u_t, \quad 0 < x < 2, t > 0;$

$u(0,t) = u(2,t) = 0 \quad t > 0;$
with $u(x,0) =$ the above $f(x)$ for $0 < x < 2$.

We already have the sine series with $L = 2$ and $\alpha = 9$. Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(9n\pi/2)^2)t} \sin((n\pi/2)x).$$

Example 10.5/7 : $100u_{xx} = u_t, \quad 0 < x < 1, t > 0;$
$u(0,t) = u(1,t) = 0 \quad t > 0;$
$u(x,0) = \sin(2\pi x) - \sin(5\pi x).$

So $\alpha = 10, L = 1$. In this case, the initial function is given as a sine series.

$$u(x,t) = e^{-(10 \cdot 2\pi)^2 t} \sin(2\pi x) - e^{-(10 \cdot 5\pi)^2 t} \sin(5\pi x)$$

$$= e^{-400\pi^2 t} \sin(2\pi x) - e^{-2500\pi^2 t} \sin(5\pi x).$$

Example 10.5/ BD10; BDM 10 :
$u_{xx} = u_t, \quad 0 < x < 40, t > 0;$
$u(0, t) = u(40, t) = 0 \quad t > 0;$

$$u(x, 0) = f(x) = \begin{cases} x & 0 \le x \le 20, \\ 40 - x & 20 < x < 40. \end{cases}$$

So $\alpha = 1, L = 40$. In this case, we need the sine series for f.

$$b_n = \frac{2}{40}\int_0^{40} f(x)\sin(\frac{n\pi}{40}x)dx =$$

$$\frac{1}{20}[\int_0^{20} x\sin(\frac{n\pi}{40}x)dx + \int_{20}^{40}(40-x)\sin(\frac{n\pi}{40}x)dx].$$

$$\int_0^{20} x\sin(\frac{n\pi}{40}x)dx = (x)(-\frac{40}{n\pi}\cos(\frac{n\pi}{40}x)) - (1)(-(\frac{40}{n\pi})^2\sin(\frac{n\pi}{40}x))|_0^{20}$$

$$= -\frac{20\cdot 40}{n\pi}\cos(\frac{n\pi}{2}) + (\frac{40}{n\pi})^2\sin(\frac{n\pi}{2}).$$

$$\int_{20}^{40}(40-x)\sin(\frac{n\pi}{40}x)dx =$$

$$(40-x)(-\frac{40}{n\pi}\cos(\frac{n\pi}{40}x)) - (-1)(-(\frac{40}{n\pi})^2\sin(\frac{n\pi}{40}x))|_{20}^{40}$$

$$= \frac{20\cdot 40}{n\pi}\cos(\frac{n\pi}{2}) + (\frac{40}{n\pi})^2\sin(\frac{n\pi}{2}).$$

The first terms in the two integrals cancel and the other two are equal.
$b_n = \frac{1}{20}[2 \cdot ((\frac{40}{n\pi})^2 \sin(\frac{n\pi}{2}))] = (\frac{160}{(n\pi)^2} \sin(\frac{n\pi}{2}))$.

$$u(x,t) = \Sigma_{n=1}^{\infty} e^{-(\frac{n\pi}{40})^2 t} \times (\frac{160}{(n\pi)^2} \sin(\frac{n\pi}{2})) \sin(\frac{n\pi}{40} x).$$

While you can leave the result as above, $\sin(\frac{n\pi}{2})) = 0$ if n is even and $= (-1)^{k-1}$ if $n = 2k - 1$.

$$u(x,t) = \Sigma_{k=1}^{\infty} e^{-(\frac{(2k-1)\pi}{40})^2 t} \times \frac{(-1)^{k-1}160}{((2k-1)\pi)^2} \sin(\frac{(2k-1)\pi}{40} x).$$

Example 10.5/ BD1; BDM1 : $xu_{xx} + u_t = 0$.
With $u = X(x)T(t)$ this becomes $xX''T + XT' = 0$
or

$$\frac{xX''}{X} = -\frac{T'}{T}.$$

With each the common constant λ we obtain the ODE's

$$T' = -\lambda T \qquad \text{and} \qquad xX'' - \lambda X = 0.$$

Example $u_{xx} + xu_{xt} - u_t = 0$.
With $u = X(x)T(t)$ this becomes $X''T + xX'T' - XT' = 0$
Factoring this is equivalent to $X''T + (xX' - X)T' = 0$
or

$$\frac{X''}{xX' - X} = -\frac{T'}{T}.$$

With each the common constant λ we obtain the ODE's

$$T' = -\lambda T \qquad \text{and} \qquad X'' - \lambda x X' + \lambda X = 0.$$