Math 39100 K (18699) - Lectures 02

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Contents

Second Order Differential Equations

A general second order ode (= ordinary differential equation) is of the form $y'' = F(x, y, y')$ or $\frac{d^2y}{dt^2} = F(t, y, \frac{dy}{dt})$.

For second order equations, we require two integrations to get back to the original function. So the *general solution* has two separate, arbitrary constants. An IVP in this case is of the form:

$$y'' = F(t, y, y'), \quad \text{with} \quad y(t_0) = y_0, \; y'(t_0) = y_0'.$$

Interpreting y' as the velocity and y'' as the acceleration, we thus determine the constants by specifying the initial position and the initial velocity with t_0 regarded as the initial time (usually $= 0$).

The Fundamental Existence and Uniqueness Theorem says that an IVP has a unique solution.

The General Solution

The *general solution* of the equation $y'' = F(t, y, y')$ is of the form $y(t, C_1, C_2)$ with two arbitrary constants. Supposing that these are solutions for every choice of C_1, C_2, how do we now that we have all the solutions? The answer is that we have all solutions if we can solve every IVP. To be precise:

THEOREM: If $y'' = F(t, y, y')$ is an equation to which the Fundamental Theorem applies and $y(t, C_1, C_2)$ is a solution for every C_1, C_2, then every solution is one of these provided that for every pair of real numbers A, B, we can find C_1, C_2 so that $y(t_0, C_1, C_2) = A$ and $y'(t_0, C_1, C_2) = B$.

PROOF: If $z(t)$ is any solution then let $A = z(t_0)$ and $B = z'(t_0)$, and choose C_1, C_2 so that $y(t, C_1, C_2)$ solves this IVP. Then $y(t, C_1, C_2)$ and $z(t)$ solve the same IVP. This means that $z(t) = y(t, C_1, C_2)$.

Linear Equations and Linear Operators

We will be studying exclusively second order, *linear* equations. These are of the form

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t), \quad \text{or} \quad y'' + py' + qy = f.$$

Just as in the first order linear case, if we see $Ay'' + By' + Cy = D$ with A, B, C, D functions of t we can obtain the above form by dividing by A. When A, B and C are constants we often won't bother.

The equation is called *homogeneous* when $f = 0$. It has *constant coefficients* when p and q are constant functions of t.

These are studied by using *linear operators*. An *operator* is a function $\mathcal{L}$ whose input and output are functions. For example, $\mathcal{L}(y) = yy'$. If $y = e^x$ then $\mathcal{L}(y) = e^{2x}$ and $\mathcal{L}(\sin)(x) = \sin(x)\cos(x)$.

An operator $\mathcal{L}$ is a linear operator when it satisfies *linearity* or *The Principle of Superposition*.

$$\mathcal{L}(Cy_1 + y_2) = C\mathcal{L}(y_1) + \mathcal{L}(y_2).$$

Given functions p, q we define $\mathcal{L}(y) = y'' + py' + qy$. Observe that:

$$\begin{array}{rcccccc}
C \times (& y_1'' & + & py_1' & + & qy_1) \\
+ & y_2'' & + & py_2' & + & qy_2 \\
\hline
\multicolumn{6}{c}{(Cy_1 + y_2)'' + p(Cy_1 + y_2)' + q(Cy_1 + y_2)}
\end{array}$$

So the homogeneous equation $y'' + py' + qy = 0$ can be written as $\mathcal{L}(y) = 0$.

General Solution of the Homogeneous Equation, Section 3.2

If y_1 and y_2 are solutions of the homogeneous equation $\mathcal{L}(y) = 0$ then by linearity $C_1 y_1 + C_2 y_2$ are solutions for every choice of constants C_1 and C_2, because $\mathcal{L}(C_1 y_1 + C_2 y_2) = C_1 \mathcal{L}(y_1) + C_2 \mathcal{L}(y_2) = C_1 0 + C_2 0 = 0.$

Notice there are two arbitrary constants. This is the general solution provided we can solve every IVP. So given numbers A, B we want to find C_1, C_2 so that

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) &= A, \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) &= B. \end{aligned}$$

Cramer's Rule says that this has a solution provided the coefficient determinant

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \quad \text{is nonzero.}$$

The Wronskian

Given two functions y_1, y_2 the *Wronskian* is defined to be the function:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

Two functions are called *linearly dependent* when there are constants C_1, C_2 not both zero such that $C_1 y_1 + C_2 y_2 \equiv 0$ and so when one of the two functions is a constant multiple of the other. If $y_2 = C y_1$ then $W(y_1, y_2) \equiv 0$. The converse is almost true:
$W/y_1^2 = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = (\frac{y_2}{y_1})'$ and so if W is identically zero the ratio $\frac{y_2}{y_1}$ is some constant C and so $y_2 = C y_1$.
On the other hand, $y_1(t) = t^3$ and $y_2(t) = |t^3|$ have $W \equiv 0$ but are not linearly dependent. What is the problem? Hint: look for a division by zero.

Now suppose the Wronskian vanishes at a single point t_0. If $y_1'(t_0) = y_2'(t_0) = 0$ then we can pick C_1, C_2, not both zero, such that with $z = C_1 y_1 + C_2 y_2$ is zero at t_0 and of course $z'(t_0) = C_1 y_1'(t_0) + C_2 y_2'(t_0) = 0$. Otherwise, choose $C_1 = y_2'(t_0)$ and $C_1 = -y_1'(t_0)$. Check that with $z = C_1 y_1 + C_2 y_2$ we have

$$z(t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) = W(y_1, y_2)(t_0) = 0,$$
$$z'(t_0) = y_1'(t_0)y_2'(t_0) - y_2'(t_0)y_1'(t_0) = 0.$$

If y_1, y_2 are solutions of the homogeneous equation $y'' + py' + qy = 0$ then $z = C_1 y_1 + C_2 y_2$ is a solution with zero initial conditions. This means that $z \equiv 0$ and so y_1 and y_2 are linearly dependent.

A pair y_1, y_2 of solutions of the homogeneous equation $y'' + py' + qy = 0$ is called a *fundamental pair* if the Wronskian does not vanish. The general solution is then $C_1 y_1 + C_2 y_2$ because we can solve every IVP.

Abel's Theorem

Abel's Theorem shows that the Wronskian of two solutions y_1, y_2 of $y'' + py' + qy = 0$ satisfies a first order linear ode:

$$W' = y_1 y_2'' - y_2 y_1'' + y_1' y_2' - y_2' y_1' = y_1 y_2'' - y_2 y_1''$$

$$\begin{array}{rcccl} q \times & y_1 y_2 & - & y_2 y_1 & = & 0 \\ p \times & y_1 y_2' & - & y_2 y_1' & = & W \\ 1 \times & y_1 y_2'' & - & y_2 y_1'' & = & W' \end{array}$$

The left side adds up to $y_1 0 - y_2 0 = 0$ and so $0 = W' + pW$ or $\frac{dW}{dt} + pW = 0$.

This is variables separable with solution $W = C \times e^{-\int p(t)dt}$.

Since the exponential is always positive, $W \equiv 0$ if $C = 0$ and otherwise W is never zero.

Homogeneous, Constant Coefficients, Sections 3.1, 3.3

The derivative itself is an example of a linear operator: $D(y) = y'$. For a linear operator $\mathcal{L}$ a function y is an *eigenvector* with *eigenvalue* r when $\mathcal{L}(y) = ry$. For the operator D, an eigenvector y satisfies $y' = D(y) = ry$. This is exponential growth with growth rate r and so the solutions are constant multiples of $y = e^{rt}$. Thus, such exponential functions have a special role to play.

Example: $y'' - y' - 6y = 0$. So that $\mathcal{L}(y) = y'' - y' - 6y$.

We can write
$\mathcal{L} = (D^2 - D - 6)(y) = (D+2)(D-3)(y) = (D-3)(D+2)(y)$.

So if $(D-3)(y) = 0$ or $(D+2)(y) = 0$, then $\mathcal{L}(y) = 0$.
$(D-3)(y) = 0$ when $y' = 3y$ and so when $y = e^{3t}$.
$(D+2)(y) = 0$ when $y = e^{-2t}$.
So $y_1 = e^{3t}$ and $y_2 = e^{-2t}$ are solutions of $y'' - y' - 6y = 0$.

A second order, linear, homogeneous equation with constant coefficients is of the form $Ay'' + By' + Cy = 0$ with A, B, C constants and $A \neq 0$.

We look for a solution of the form $y = e^{rt}$. Substituting and factoring we get

$$(Ar^2 + Br + C)e^{rt} = 0.$$

Since the exponential is always positive, e^{rt} is a solution precisely when r satisfies the characteristic equation $Ar^2 + Br + C = 0$.

The quadratic equation $Ar^2 + Br + C = 0$ has roots

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

For the equation $Ar^2 + Br + C = 0$ the nature of the two roots

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

depends on the discriminant $B^2 - 4AC$.

CASE 1: The simplest case is when $B^2 - 4AC > 0$. There are then two distinct real roots r_1 and r_2. This gives us two special solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ with the Wronskian.

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0.$$

The general solution is then $C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

CASE 2: If $B^2 - 4AC < 0$ then the roots are a complex conjugate pair $a \pm \mathbf{i} b = \frac{-B}{2A} \pm \mathbf{i} \frac{\sqrt{|B^2 - 4AC|}}{2A}$. This requires a consideration of complex numbers.

Polar Form of Complex Numbers; The Euler Identity

We represent the complex number $z = a + \mathbf{i}b$ as a vector in $\mathbb{R}^2$ with x-coordinate a and y-coordinate b. Addition and multiplication by real scalars is done just as with vectors. For multiplication $(a + \mathbf{i}b)(c + \mathbf{i}d) = (ac - bd) + \mathbf{i}(ad + bc)$. The *conjugate* is $\bar{z} = a - b\mathbf{i}$ and $z\bar{z} = a^2 + b^2 = |z|^2$ which is positive unless $z = 0$.

Multiplication is also dealt with by using the *polar* form of the complex number. This requires a bit of review.

Recall from Math 203 the three important Maclaurin series:

$$
\begin{aligned}
e^t &= 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{t^7}{7!} + \ldots \\
\cos(t) &= 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots \\
\sin(t) &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots
\end{aligned}
$$

The exponential function is the first important example of a function f on $\mathbb{R}$ which is neither *even* with $f(-t) = f(t)$ nor *odd* with $f(-t) = -f(t)$ nor a mixture of even and odd functions in an obvious way like a polynomial. It turns out that any function f on $\mathbb{R}$ can be written as the sum of an even and an odd function by writing

$$f(t) \quad = \quad \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2}.$$

You have already seen this for $f(t) = e^t$

$$\begin{aligned}
e^{-t} &= 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \dots \\
\cosh(t) &= \frac{e^t + e^{-t}}{2} = 1 + \frac{t^2}{2} + \frac{t^4}{4!} + \frac{t^6}{6!} + \dots \\
\sinh(t) &= \frac{e^t - e^{-t}}{2} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \dots
\end{aligned}$$

Now we use the same trick but with $e^{\mathbf{i}t}$

$$\begin{aligned} e^{\mathbf{i}t} &= 1 + \mathbf{i}t - \frac{t^2}{2} - \mathbf{i}\frac{t^3}{3!} + \frac{t^4}{4!} + \mathbf{i}\frac{t^5}{5!} - \frac{t^6}{6!} - \mathbf{i}\frac{t^7}{7!} + \ldots \\ e^{-\mathbf{i}t} &= 1 - \mathbf{i}t - \frac{t^2}{2} + \mathbf{i}\frac{t^3}{3!} + \frac{t^4}{4!} - \mathbf{i}\frac{t^5}{5!} - \frac{t^6}{6!} + \mathbf{i}\frac{t^7}{7!} + \ldots \\ \frac{e^{\mathbf{i}t} + e^{-\mathbf{i}t}}{2} &= 1 \quad - \frac{t^2}{2} \quad + \frac{t^4}{4!} \quad - \frac{t^6}{6!} + \ldots \\ \frac{e^{\mathbf{i}t} - e^{-\mathbf{i}t}}{2} &= \mathbf{i}[t \quad - \frac{t^3}{3!} \quad + \frac{t^5}{5!} \quad - \frac{t^7}{7!} + \ldots] \end{aligned}$$

So the even part of $e^{\mathbf{i}t}$ is $\cos(t)$ and the odd part is $\mathbf{i}\sin(t)$. Adding them we obtain *Euler′sIdentity* and its conjugate version.

$$\begin{aligned} e^{\mathbf{i}t} &= \cos(t) + \mathbf{i}\sin(t) \\ e^{-\mathbf{i}t} &= \cos(t) - \mathbf{i}\sin(t) \end{aligned}$$

Substituting $t = \pi$ we obtain $e^{\mathbf{i}\pi} = -1$ and so $e^{\mathbf{i}\pi} + 1 = 0$.

Now we start with a complex number z in rectangular form $z = x + \mathbf{i}y$ and convert to polar coordinates with $x = r\cos(\theta), y = r\sin(\theta)$ so that $r^2 = x^2 + y^2 = z\bar{z}$. The length r is called the *magnitude* of z. The angle θ is called the *argument* of z. We obtain

$$z \quad = \quad x \, + \, \mathbf{i}y \quad = \quad r(\cos(\theta) \, + \, \mathbf{i}\sin(\theta)) \quad = \quad re^{\mathbf{i}\theta}.$$

Now we return to CASE 2 with the pair of roots $r = a \pm \mathbf{i}b$. We obtain two complex solutions

$$\begin{aligned} e^{(a+\mathbf{i}b)t} &= e^{at}(\cos(bt) + \mathbf{i}\sin(bt)), \\ e^{(a-\mathbf{i}b)t} &= e^{at}(\cos(bt) - \mathbf{i}\sin(bt)) \end{aligned}$$

When we add these two solutions and divide by 2 we obtain the real solution $e^{at}\cos(bt)$ When we subtract and divide by $2\mathbf{i}$ we obtain the real solution $e^{at}\sin(bt)$.

Thus, when the roots are $r = a \pm \mathbf{i}b$, we have the two solutions $y_1 = e^{at}\cos(bt)$ and $y_2 = e^{at}\sin(bt)$ with the Wronskian

$$W = \begin{vmatrix} e^{at}\cos(bt) & e^{at}\sin(bt) \\ ae^{at}\cos(bt) - be^{at}\sin(bt) & ae^{at}\sin(bt) + be^{at}\cos(bt) \end{vmatrix}$$

This is be^{2at} which is not zero because the imaginary part b is not zero. Thus we have so far:

CASE 1 ($B^2 - 4AC > 0$, Two distinct real roots r_1, r_2) The general solution is $C_1e^{r_1t} + C_2e^{r_2t}$

CASE 2 ($B^2 - 4AC < 0$, Complex conjugate pair of roots $a \pm \mathbf{i}b$) The general solution is $C_1e^{at}\cos(bt) + C_2e^{at}\sin(bt)$.

Example 3.3/ BD 19, BDM 13: $y'' - 2y' + 5y = 0$, $y(\pi/2) = 0$, $y'(\pi/2) = 3$.

The characteristic equation is $r^2 - 2r + 5 = 0$ with roots $r = \frac{+2 \pm \sqrt{4-20}}{2} = 1 \pm 2\mathbf{i}$.

So the general solution is:

$$y = C_1 e^t \cos(2t) + C_2 e^t \sin(2t).$$

When $t = \pi/2, \sin(2t) = 0, \cos(2t) = -1$.

$$\begin{aligned} y &= C_1 e^t \cos(2t) + C_2 e^t \sin(2t), \\ y' &= C_1[e^t \cos(2t) - 2e^t \sin(2t)] + \\ &\qquad C_2[e^t \sin(2t) + 2e^t \cos(2t)]. \end{aligned}$$

At $t = \pi/2, \quad 0 = C_1[-e^{\pi/2}] + C_2[0]$,
$3 = C_1[-e^{\pi/2}] + C_2[-2e^{\pi/2}]$.
So $C_1 = 0$, $C_2 = -(3/2)e^{-\pi/2}$ and so
$y = -(3/2)e^{-\pi/2} e^t \sin(2t)$.

We looked at $y'' - 3y = 0$ which had, we saw, general solution

$$y = C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t}.$$

Now look at $y'' + 3y = 0$ with characteristic equation
$r^2 + 3 = 0$.
It has conjugate roots $\pm\sqrt{3}\mathbf{i}$ and so the general solution is:

$$y = C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t).$$

Reduction of Order, Section 3.4

There remains CASE 3: If $B^2 - 4AC = 0$ there is a single (real) root $r = \frac{-B}{2A}$, "repeated twice". So we have so far in that case only one solution e^{rt} and we need a second independent solution.

For a general second order, linear homogeneous equation $y'' + py' + qy = 0$ with constant coefficients or not, we need two independent solutions y_1, y_2 to get the general solution $C_1 y_1 + C_2 y_2$.

Suppose we have one solution y_1. We look for another solution of the form $y_2 = uy_1$, with u not a constant. Substitute:

$$\begin{array}{rcccccc} q \times & y_2 & = & uy_1 & & & \\ p \times & y_2' & = & uy_1' & + & u'y_1 & \\ 1 \times & y_2'' & = & uy_1'' & + & 2u'y_1' & + \quad u''y_1 \end{array}$$

$$0 = 0 + u'(py_1 + 2y_1') + u''y_1$$

The equation $(y_1)u'' + (py_1 + 2y_1')u' = 0$ has no variable u and so by letting $v = u', v' = u''$ we obtain the first order equation $(y_1)v' + (py_1 + 2y_1')v = 0$ which variables separable with solution $v = exp(-\int \ p + 2(y_1'/y_1))$. Notice that $u' = v$ is not zero and so when we integrate to get u it is not a constant and so $y_2 = uy_1$ together with y_1 forms a fundamental pair of solutions.

Now we return to CASE 3 where the quadratic equation $Ar^2 + Br + C = 0$ has a repeated root $r = r^*$. This means that $Ar^2 + Br + C = A(r - r^*)^2$ and so $B = -2Ar^*$ and $C = A(r^*)^2$.

The ode is $y'' - 2r^*y' + (r^*)^2y = 0$ with a solution $y_1 = e^{r^*t}$. We look for $y_2 = ue^{r^*t}$.

$$
\begin{array}{rrcll}
(r^*)^2 \times & y_2 & = & ue^{r^*t} & \\
-2r^* \times & y_2' & = & ur^*e^{r^*t} & + u'e^{r^*t} \\
1 \times & y_2'' & = & u(r^*)^2e^{r^*t+2u'r^*e^{r^*t}+u''e^{r^*t}} &
\end{array}
$$

$$
0 \quad = \quad 0 \quad + \quad 0 \quad + \quad u''e^{r^*t}
$$

So $v' = u'' = 0, v = 1$ and $u = t$. So the general solution for CASE 3 is $C_1e^{r^*t} + C_2te^{r^*t}$.

This completes the story for second order, linear homogeneous equations with constant coeeficients.

CASE 1 ($B^2 - 4AC > 0$, Two distinct real roots r_1, r_2) The general solution is $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

CASE 2 ($B^2 - 4AC < 0$, Complex conjugate pair of roots $a \pm \mathbf{i} b$) The general solution is $C_1 e^{at} \cos(bt) + C_2 e^{at} \sin(bt)$.

CASE 3 ($B^2 - 4AC = 0$, single repeated, real root r^*) The general solution is $C_1 e^{r^* t} + C_2 t e^{r^* t}$.

In CASE 3, we used the method of Reduction of Order, but we simply write down the solution. However, for nonconstant coefficients we will require the method itself.

Example 3.4/ BD11, BDM9 :
$9y'' - 12y' + 4y = 0, \quad y(0) = 2, y'(0) = -1$

Characteristic Equation $0 = 9r^2 - 12r + 4 = (3r - 2)(3r - 2)$
$r = 2/3$ repeated twice.

The general solution is

$$y \;=\; C_1 e^{(2/3)t} + C_2 t e^{(2/3)t}.$$

$$
\begin{aligned}
y &= C_1 e^{(2/3)t} \quad + \quad C_2 t e^{(2/3)t}, \\
y' &= [(2/3)C_1 + C_2]e^{(2/3)t} + (2/3)C_2 t e^{(2/3)t},
\end{aligned}
$$

$y(0) = 2, y'(0) = -1$ and so

$$2 = C_1, \quad -1 = (2/3)C_1 + C_2,$$

And so $-(7/3) = C_2$.

The solution is $y = [2 - (7/3)t]e^{(2/3)t}$.

Example 3.4/BD28 : $(x-1)y'' - xy' + y = 0, x > 1$ with $y_1(x) = e^x$.

$$\begin{array}{rcccccccc}
1 \times & y_2 & = & ue^x & & & & & \\
-x \times & y_2' & = & ue^x & + & u'e^x & & & \\
(x-1) \times & y_2'' & = & ue^x & + & 2u'e^x & + & u''e^x & \\
\end{array}$$

$$0 = 0 + u'(x-2)e^x + u''(x-1)e^x$$

$(x-1)v' = -(x-2)v$, and so $\int \frac{dv}{v} = \int -1 + \frac{1}{x-1} dx$.

$u' = v = (x-1)e^{-x}$, and so $u = -xe^{-x}$ and $y_2 = uy_1 = -x$.

Using the Wronskian

As one student pointed out, there is an alternative way of obtaining a second solution by using Abel's Theorem. Recall that it says that if W is the Wronskian for a pair of solutions y_1, y_2 of the second order, homogeneous, linear equation $y'' + py' + qy = 0$ then

$$W \;=\; C \cdot exp[-\int p(t)dt].$$

Suppose we have a solution y_1. We look for a second solution y_2 such that $W(y_1, y_2) = exp[-\int p(t)dt]$. We choose $C = 1$ because we are looking for a single solution y_2. So y_2 satisfies the first order linear equation

$$y_1 y_2' - y_1' y_2 = exp[-\int p(t)dt].$$

As an example the student chose: $x^2 y'' - 7xy' + 16y = 0$ with $y_1 = x^4$.

$$
\begin{aligned}
16 \times \; y_2 &= ux^4 \\
-7x \times \; y_2' &= u4x^3 \quad + \; u'x^4 \\
x^2 \times \; y_2'' &= u12x^2 \quad + \; u'8x^3 \; + \; u''x^4
\end{aligned}
$$

$$
0 \quad = \; 0 + u'x^5 \; + \; u''x^6
$$

$x^6 v' = -x^5 v$, and so $\int \frac{dv}{v} = \int -\frac{dx}{x}$.

$\ln v = -\ln x$ and so $u' = v = x^{-1}$,

$$
u = \ln x, \qquad y_2 = uy_1 = x^4 \ln x
$$

.

Instead we use Abel's Theorem applied to: $y'' - \frac{7}{x}y' + \frac{16}{x^2}y = 0$.

$$x^4 y_2' - 4x^3 y_2 = exp[7 \ln x] = x^7,$$
$$y_2' - \frac{4}{x} y_2 = x^3.$$

$$\mu = x^{-4}, [x^{-4} y_2]' = x^{-1}.$$

$$x^{-4} y_2 = \ln x, \qquad y_2 = x^4 \ln x.$$

Second Order Linear Nonhomogeneous Equations

Consider now a second order linear equation which is not homogeneous $y'' + py' + qy = f$ where p, q and f are functions of t.

Recall the linear operator $\mathcal{L}(y) = y'' + py' + qy$. To solve the homogeneous equation $\mathcal{L}(y) = 0$ we saw that we need a pair of independent solutions y_1, y_2. Then the general solution of the homogeneous equation is $y_h = C_1 y_1 + C_2 y_2$. To solve the equation $\mathcal{L}(y) = f$ we need just one solution, *particular solution* y_p so that $\mathcal{L}(y_p) = f$. Then the general solution is $y_g = y_h + y_p = C_1 y_1 + C_2 y_2 + y_p$. Notice first

$$\mathcal{L}(C_1 y_1 + C_2 y_2 + y_p) = C_1 \mathcal{L}(y_1) + C_2 \mathcal{L}(y_2) + \mathcal{L}(y_p) = 0 + 0 + f = f.$$

So for every choice of C_1 and C_2, y_g is a solution.

To show that by varying the two arbitrary constants y_g gives us all solutions, it is enough to show that we can solve every initial value problem. That is, given A, B we have to choose C_1, C_2 to satisfy the initial conditions $y_g(t_0) = A, y_g'(t_0) = B$. This means that we want to solve:

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) + y_p(t_0) &= A, \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) + y_p'(t_0) &= B. \end{aligned}$$

But this is the same as:

$$\begin{aligned} C_1 y_1(t_0) + C_2 y_2(t_0) &= A - y_p(t_0), \\ C_1 y_1'(t_0) + C_2 y_2'(t_0) &= B - y_p'(t_0). \end{aligned}$$

This has a solution because the Wronskian $W(y_1, y_2)(t_0) \neq 0$. Since every IVP has a solution of the form y_g, we have all solutions.

Nonhomogeneous Equations: Undetermined Coefficients, Section 3.5

Our first method for finding the particular solution applies when the homogeneous equation $Ay'' + By' + Cy = 0$ has constant coefficients the *forcing function* is a sum so that each term has an associated root according to the following table.

$$\begin{array}{rcl} poly(t) & \Rightarrow & \text{root equals } 0, \\ poly(t)e^{rt} & \Rightarrow & \text{root equals } r, \\ \begin{array}{c} poly(t)e^{at}\cos(bt) \\ poly(t)e^{at}\sin(bt) \end{array} & \Rightarrow & \text{root equals } a \pm \mathbf{i}b. \end{array}$$

The key is that when you take the derivative of expressions associated with a particular root (or conjugate root pair) you get expressions associated with the same root and with the polynomial in front of no higher degree.

To solve $Ay'' + By' + Cy = f$ with forcing function $f(t)$ we find the *Test Function (First Version)* $Y_p^1(t)$ in two steps:

STEP 1: Find the associated root for each term and group according to the associated roots.

STEP 2a: For the real associated root r (including $r = 0$), you use

$$(A_n t^n + A_{n-1} t^{n-1} + \dots A_1 t + A_0) e^{rt}$$

where n is the highest power of t attached to any e^{rt} in $r(t)$.

STEP 2b: For the complex pair $a \pm \mathbf{i}b$, you use

$$(A_n t^n + A_{n-1} t^{n-1} + \dots A_1 t + A_0) e^{at} \cos(bt) +$$
$$(B_n t^n + B_{n-1} t^{n-1} + \dots B_1 t + B_0) e^{at} \sin(bt)$$

where n is the highest power of t attached to any $e^{at} \cos(bt)$ or $e^{at} \sin(bt)$ in $r(t)$.

IMPORTANT: You have to include both the sines and the cosines with that highest power applied to both.

To get the final version of the test function we need a preliminary step.

STEP 0: Solve the homogeneous equation $Ay'' + By' + Cy = 0$. In the process, you get the roots of the characteristic equation $Ar^2 + Br + C = 0$.

STEP 3: If a root from the homogeneous equation occurs among the associated roots then the corresponding expression in Y_p^1 is multiplied by the just high enough power of t so that the sum includes no solution of the homogeneous equation. After this adjustment we have the final Test Function Y_p.

STEP 4: Substitute Y_p into the equation and equate coefficients to determine the unknown coefficients to obtain the particular solution y_p.

The general solution is then $y_g = y_h + y_p$. If there are initial conditions, then substitute to determine C_1 and C_2 after y_p has been obtained.

Example: $f(t) =$
$t^3 + 2t - 5 + 7\cos(3t) - (t^2+1)e^{2t} - e^{2t}\sin(3t) - t^2\sin(3t)$.

STEP 1: The associated roots are as follows:

$$\begin{aligned} t^3 + 2t - 5 \quad &\Rightarrow \quad \text{root equals } 0, \\ 7\cos(3t) - t^2\sin(3t) \quad &\Rightarrow \quad \text{root equals } 0 \pm 3\mathbf{i}, \\ -(t^2+1)e^{2t} \quad &\Rightarrow \quad \text{root equals } 2, \\ -e^{2t}\sin(3t) \quad &\Rightarrow \quad \text{root equals } 2 \pm 3\mathbf{i}. \end{aligned}$$

STEP 2: The first version of the test function, $Y_p^1(t)$ is

$$\begin{gathered} [At^3 + Bt^2 + Ct + D] + \\ [(Et^2 + Ft + G)\cos(3t) + (Ht^2 + It + J)\sin(3t)] + \\ [(Kt^2 + Lt + M)e^{2t}] + \\ [Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)]. \end{gathered}$$

The final form of Y_p depends on the left side of the equation. Example A: $y'' - 2y' = f(t)$, with characteristic equation $r^2 - 2r = 0$ and so with roots $0, 2$.
STEP 0: $y_h = C_1 + C_2 e^{2t}$.
STEP 3: In Y_p^1 the terms D and Me^{2t} are solutions of the homogeneous equation. When we apply $\mathcal{L}(y) = y'' - 3y'$, these terms drop out, with D and M gone. For the remaining 13 unknowns we get 15 equations and usually no solution. We multiply each of the corresponding blocks by t so that Y_p is

$$[t(At^3 + Bt^2 + Ct + D)]+$$
$$[(Et^2 + Ft + G)\cos(3t) + (Ht^2 + It + J)\sin(3t)]+$$
$$[t(Kt^2 + Lt + M)e^{2t}]+$$
$$[Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)].$$

STEP 4: Substitute Y_p into the equation $\mathcal{L}(y) = f$ and solve for the 15 unknowns to obtain the particular solution y_p. The general solution is then $y_g = y_h + y_p$.

Example B: $y'' - 4y' + 4y = f(t)$, with characteristic equation $r^2 - 4r + 4 = 0$ and so with roots $2, 2$.

STEP 0: $y_h = C_1 e^{2t} + C_2 t e^{2t}$.

STEP 3: In Y_p^1 the terms $Lte^{2t} + Me^{2t}$ are solutions of the homogeneous equation. Multiplying by t will not suffice as we then obtain Mte^{2t} which is still a solution of the homogeneous equation. Multiply by t again to get for Y_p:

$$
\begin{gathered}
[At^3 + Bt^2 + Ct + D]+ \\
[(Et^2 + Ft + G)\cos(3t) + (Ht^2 + It + J)\sin(3t)]+ \\
[t^2(Kt^2 + Lt + M)e^{2t}]+ \\
[Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)].
\end{gathered}
$$

Example C: $y'' + 9y = f(t)$ with characteristic equation $r^2 + 9 = 0$ and so with roots $\pm 3\mathbf{i}$.

STEP 0: $y_h = C_1 \cos(3t) + C_2 \sin(3t)$.

STEP 3: Now it is $G\cos(3t) + J\sin(3t)$ which case the problem. Multiplying by t we get for Y_p

$$
\begin{gathered}
[At^3 + Bt^2 + Ct + D] + \\
[t(Et^2 + Ft + G)\cos(3t) + t(Ht^2 + It + J)\sin(3t)] + \\
[(Kt^2 + Lt + M)e^{2t}] + \\
[Ne^{2t}\cos(3t) + Pe^{2t}\sin(3t)].
\end{gathered}
$$

Question:

$$\begin{cases} (a) & y'' - y' - 2y \\ (b) & y'' - y' + 2y \qquad = \\ (c) & 4y'' - 4y' + y \end{cases}$$

$$t^2 + 7te^{2t} - e^{t/2}\sin((\sqrt{7}/2)t) - t^2e^{t/2} + 1.$$

(0) Solve each of the three homogeneous equations.
(1) Block together the terms with the same associated root.
(2) Obtain the first version of the test function $Y^1(t)$ by using the highest power ot t for each block. Remember to include both sines and cosines.
(3) Adjust any block whose associated root is a root of the homogeneous equation to get the test function $Y(t)$.

(0 - a) Characteristic equation $r^2 - r - 2 = (r-2)(r+1) = 0$ with roots $2, -1$. So $y_h = C_1 e^{2t} + C_2 e^{-t}$.
(0 - b) Characteristic equation $r^2 - r + 2 = 0$ with roots $\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathbf{i}$.
So $y_h = C_1 e^{t/2} \cos((\sqrt{7}/2)t) + C_2 e^{t/2} \sin((\sqrt{7}/2)t)$.
(0 - c) Characteristic equation
$4r^2 - 4r + 1 = (2r-1)(2r-1) = 0$ with roots $\frac{1}{2}, \frac{1}{2}$
So $y_h = C_1 e^{t/2} + C_2 t e^{t/2}$.

(1) $[t^2+1]$ associated root 0.

$7te^{2t}$ associated root 2.

$-e^{t/2}\sin((\sqrt{7}/2)t)$ associated roots $\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathbf{i}$.

$-t^2e^{t/2}$ associated root $\frac{1}{2}$.

$$
\begin{aligned}
&Y^1(t) = [At^2 + Bt + C] \\
&+[(Dt + E)e^{2t}] \\
&+[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] \\
&+[(Ht^2 + It + J)e^{t/2}].
\end{aligned}
$$

For equation (a) the root 2 is a root of the homogeneous equation and so
$Y(t) = [At^2 + Bt + C] + t[(Dt + E)e^{2t}]$
$+[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] + [(Ht^2 + It + J)e^{t/2}]$.

For equation (b) the roots $\frac{1}{2} \pm \frac{\sqrt{7}}{2}\mathbf{i}$. are roots of the homogeneous equation and so
$Y(t) = [At^2 + Bt + C] + [(Dt + E)e^{2t}]$
$+t[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] + [(Ht^2 + It + J)e^{t/2}]$.

For equation (c) the root $\frac{1}{2}$ is a repeated root of the homogeneous equation and so
$Y(t) = [At^2 + Bt + C] + [(Dt + E)e^{2t}]$
$+[Fe^{t/2}\cos((\sqrt{7}/2)t) + Ge^{t/2}\sin((\sqrt{7}/2)t)] + t^2[(Ht^2 + It + J)e^{t/2}]$.

Exercise 3.5/ BD19; BDM14 :
$y'' + 4y = 3\sin 2t, \qquad y(0) = 2, y'(0) = -1.$
STEP 0: Roots for the homogeneous are $\pm 2\mathbf{i}$. So
$y_h = C_1\cos(2t) + C_2\sin(2t)$.
STEP 1: Associated root for $3\sin 2t$ is $\pm 2\mathbf{i}$ and so STEP 2:
$Y_p^1 = A\cos(2t) + B\sin(2t)$.
STEP 3: $Y_p = At\cos(2t) + Bt\sin(2t)$, because the associated roots are roots of the homogeneous equation.
STEP 4: Substitute

$$
\begin{array}{rl}
4\times \ Y_p = & At\cos(2t) \ + \ Bt\sin(2t) \\
0\times \ Y_p' = & 2Bt\cos(2t) - 2At\sin(2t) + A\cos(2t) + B\sin(2t) \\
1\times \ Y_p'' = & -4At\cos(2t) - 4Bt\sin(2t) + 4B\cos(2t) - 4A\sin(2t) \\
\hline
3\sin 2t \ = & 0 \quad + \quad 0 \quad + \quad 4B\cos(2t) - 4A\sin(2t).
\end{array}
$$

So $B = 0, A = -3/4$ and
$y_g = C_1\cos(2t) + C_2\sin(2t) - \frac{3}{4}t\cos(2t)$.

Now for the initial conditions:

$$y_g = C_1\cos(2t) \;+\; C_2\sin(2t) - \frac{3}{4}t\cos(2t) + 0t\sin(2t),$$

$$y_g' = (2C_2 - \frac{3}{4})\cos(2t) - 2C_1\sin(2t) + 0t\cos(2t) + \frac{3}{4}t\sin(2t).$$

Substitute $t = 0$.

$$\begin{aligned} 2 &= y_g(0) = C_1, \\ -1 &= y_g'(0) = 2C_2 - \frac{3}{4}. \end{aligned}$$

So $C_1 = 2$, $C_2 = -\frac{1}{8}$ and

$$y = 2\cos(2t) - \frac{1}{8}\sin(2t) - \frac{3}{4}t\cos(2t).$$

Illustration

As an illustration of the Method of Undetermined Coefficients, we consider the integral problems from Calculus 2:

$$\int e^{ax} \cos bx \; dx, \quad \int e^{ax} \sin bx \; dx.$$

These were done there by a looping pair of integrations by parts. Instead think of these are the solutions to the first order linear equations with constant coefficients:

$$y' = e^{ax} \cos bx, \qquad y' = e^{ax} \sin bx, \quad (b \neq 0)$$

which we will solve using the Method of Undetermined Coefficients

$$y = Ae^{ax}\cos bx + Be^{ax}\sin bx$$
$$y' = (Aa + Bb)e^{ax}\cos bx + (-Ab + Ba)e^{ax}\sin bx.$$

Equating coefficients between y' and $1e^{ax}\cos bx$ we get

$$\begin{array}{rcrcl} Aa & + & Bb & = & 1 \\ -Ab & + & Ba & = & 0. \end{array}$$

Solve using Cramer's Rule with coefficient determinant $a^2 + b^2$

$A = a/(a^2 + b^2)$, $B = b/(a^2 + b^2)$ and so

$$\int e^{ax} \cos bx dx = \frac{ae^{ax} \cos bx \ + \ be^{ax} \sin bx}{a^2 + b^2}$$

Similarly, using

$$\begin{aligned} Aa \ + \ Bb \ &= \ 0 \\ -Ab \ + \ Ba \ &= \ 1, \end{aligned}$$

we obtain

$$\int e^{ax} \sin bx dx = \frac{-be^{ax} \cos bx + ae^{ax} \sin bx}{a^2 + b^2}.$$

For those who have had linear algebra:
The functions $\{e^{ax}\cos bx, e^{ax}\sin bx\}$ form a basis for a two dimensional vector space of functions.
On it the derivative map D is a linear map with matrix

$$D = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

That is, the first and second columns are the coefficients of D applied to $e^{ax}\cos bx$ and $e^{ax}\sin bx$, respectively.

The inverse of D is the integral map I with inverse matrix:

$$I = \frac{1}{a^2 + b^2} \cdot \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with the first and second columns the coefficients of I applied to $e^{ax} \cos bx$ and $e^{ax} \sin bx$, respectively.

Higher Order, Homogeneous, Linear Equations with Constant Coefficients, Section 4.2

For $n = 3, 4, \ldots$ an n^{th} order linear equation with constant coefficients is of the form

$$A_n y^{(n)} + A_{n-1} y^{(n-1)} + \cdots + A_1 y' + A_0 y = f.$$

It is homogeneous when $f = 0$. We will consider just the homogeneous case and the cases with associated roots so that the method of Undetermined Coefficients can be used.

For an n^{th} order equation the general solution will have n arbitrary constants. In the linear case, we look for n independent solutions $y_1, \ldots, y_n$ and get the general solution of the homogeneous equation $y_h = C_1 y_1 + \cdots + C_n y_n$.

As in the second order case, we look for solutions of the form e^{rt}. When we substitute and divide away the common factor of e^{rt} we obtain the n^{th} degree characteristic equation $P(r) = 0$, with

$$P(r) = A_n r^n + A_{n-1} r^{n-1} + \cdots + A_1 r + A_0.$$

The equation $P(r) = 0$ has, in general, n roots obtained by factoring. There are two CASES.

Real roots ($r = a$): The root $r = a$ from the factor $r - a$ gives a solution e^{at}. "Repeated roots" really means repeated factors. If we have $(r-a)^k$ in the characteristic polynomial $P(r)$, that is, k copies of $(r - a)$ then, as in the second order case, the additional solutions are obtained by multiplying by t, to get solutions $e^{at}, te^{at}, \ldots, t^{k-1}e^{at}$.

Complex conjugate pairs ($r = a \pm b\mathbf{i}$): The conjugate pair $r = a \pm b\mathbf{i}$ comes from a quadratic factor $r^2 - 2ar + (a^2 + b^2)$ gives the two solutions $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$. But now you can have repeated complex roots. If $(r^2 - 2ar + (a^2 + b^2))^k$ occurs in the factoring of $P(r)$ then we have $2k$ solutions $e^{at}\cos(bt), te^{at}\cos(bt), \ldots, t^{k-1}e^{at}\cos(bt)$ and $e^{at}\sin(bt), te^{at}\sin(bt), \ldots, t^{k-1}e^{at}\sin(bt)$.

The hard work here is doing the factoring. So we have to review (or for some of you introduce) some factoring methods.

Example A: $y^{(11)} - 2y^{(7)} + y^{(3)} = 0$ with

$$P(r) = r^{11} - 2r^7 + r^3.$$

First, pull out the common factor $P(r) = r^3(r^8 - 2r^4 + 1)$.

Next notice that $r^8 - 2r^4 + 1$ is quadratic in r^4 and factors to $(r^4 - 1)^2$. Now use the *difference of two squares* $r^4 - 1 = (r^2 + 1)(r^2 - 1) = (r^2 + 1)(r + 1)(r - 1)$. So

$$P(r) = r^3(r^2 + 1)^2(r + 1)^2(r - 1)^2$$

with eleven roots $\quad 0, 0, 0, \pm\mathbf{i}, \pm\mathbf{i}, -1, -1, +1, +1.$

$$y_h = C_1 + C_2 t + C_3 t^2 + C_4 \cos(t) + C_5 t \cos(t) + \\ C_6 \sin(t) + C_7 t \sin(t) + C_8 e^{-t} + C_9 t e^{-t} + C_{10} e^t + C_{11} t e^t.$$

Example B: $y^{(5)} - 3y^{(3)} - 8y'' + 24y = 0$ with

$P(r) = r^5 - 3r^3 - 8r^2 + 24 = (r^2 - 3)r^3 - (r^2 - 3)8 = (r^2 - 3)(r^3 - 8)$.

This is *factoring by grouping*. The roots of $r^2 - 3 = 0$ are $\pm\sqrt{3}$. For the cubic we need the *difference of two cubes*

$$r^3 - a^3 = (r - a)(r^2 + ar + a^2),$$
$$r^3 + a^3 = r^3 - (-a)^3 = (r + a)(r^2 - ar + a^2).$$

So $P(r) = (r - \sqrt{3})(r + \sqrt{3})(r - 2)(r^2 + 2r + 4)$ with roots $\sqrt{3}, -\sqrt{3}, 2, -1 \pm \sqrt{3}\mathbf{i}$ and the solution is

$$y_h = C_1 e^{\sqrt{3}t} + C_2 e^{-\sqrt{3}t} + C_3 e^{2t} + C_4 e^{-t}\cos(\sqrt{3}t) + C_5 e^{-t}\sin(\sqrt{3}t).$$

DeMoivre's Theorem

Remember that if we start with a complex number z in rectangular form $z = x + \mathbf{i}y$ and convert to polar coordinates with $x = r\cos(\theta), y = r\sin(\theta)$. The length r is called the *magnitude* and the angle θ is called the *argument* of the complex number. So we have

$$z = x + \mathbf{i}y = r(\cos(\theta) + \mathbf{i}\sin(\theta)) = re^{\mathbf{i}\theta}.$$

If $z = re^{\mathbf{i}\theta}$ and $w = ae^{\mathbf{i}\phi}$ then, by using properties of the exponential, we see that $zw = rae^{\mathbf{i}(\theta+\phi)}$. That is, the magnitudes multiply and the arguments add. In particular, for $n = 1, 2, 3 \ldots,$ we see that $z^n = r^n e^{\mathbf{i}n\theta}$.

DeMoivre's Theorem describes the solutions of the equation $z^n = a$ which has n distinct solutions when a is nonzero. We describe these solutions when a is a nonzero real number.

Notice that $re^{\mathbf{i}\theta} = re^{\mathbf{i}(\theta+2\pi)}$. When we raise this equation to the power n the angles are replaced by $n\theta$ and $n\theta + n2\pi$ and so they differ by a multiple of π. But when we divide by n we get different angles. So if $a > 0$, then $a = ae^{\mathbf{i}0}$ and $-a = ae^{\mathbf{i}\pi}$.

$$z^n = a \ \Rightarrow \ z = a^{1/n} \cdot \{e^{\mathbf{i}0}, e^{\mathbf{i}2\pi/n}, e^{\mathbf{i}4\pi/n}, .., e^{\mathbf{i}(n-1)2\pi/n}\}$$

$$z^n = -a \Rightarrow z = a^{1/n} \cdot \{e^{\mathbf{i}\pi/n}, e^{\mathbf{i}(\pi+2\pi)/n}, e^{\mathbf{i}(\pi+4\pi)/n}, .., e^{\mathbf{i}(\pi+(n-1)2\pi)/n}\}$$

For $z^n = a$ we start at angle 0 and go around the circle of radius $a^{1/n}$ with n equal steps of $2\pi/n$ radians for each step. For $z^n = -a$ we again go around the circle with n equal steps but this time starting a half step up at angle π/n.

Example C: $y^{(6)} - 64y = 0$, with $P(r) = r^6 - 2^6$. By DeMoivre's Theorem the roots are $2\{1, e^{\mathbf{i}\pi/3}, e^{\mathbf{i}2\pi/3}, -1, e^{\mathbf{i}4\pi/3}, e^{\mathbf{i}5\pi/3}\}$.

$2e^{\mathbf{i}\pi/3}, 2e^{\mathbf{i}5\pi/3}$ is the conjugate pair $1 \pm \mathbf{i}\sqrt{3}$ and $2e^{\mathbf{i}2\pi/3}, 2e^{\mathbf{i}4\pi/3}$ is the conjugate pair $-1 \pm \mathbf{i}\sqrt{3}$.

So the roots of $P(r) = 0$ are $2, -2, 1 \pm \mathbf{i}\sqrt{3}, -1 \pm \mathbf{i}\sqrt{3}$.

In this case, one can also factor using the difference of two squares and difference of two cubes:

$$r^6 - 64 = (r^3 - 8)(r^3 + 8) = \\ (r - 2)(r^2 + 2r + 4)(r + 2)(r^2 - 2r + 4),$$

$$y_h = C_1 e^{2t} + C_2 e^{-2t} + \\ C_3 e^t \cos(\sqrt{3}t) + C_4 e^t \sin(\sqrt{3}t) + C_5 e^{-t} \cos(\sqrt{3}t) + C_6 e^{-t} \sin(\sqrt{3}t)$$

Example D: $y^{(6)} + 81y'' = 0$ with
$P(r) = r^6 + 81r^2 = r^2(r^4 + 81)$.

By DeMoivre's Theorem the roots of $r^4 + 3^4 = 0$ are

$$3\{e^{\mathbf{i}\pi/4}, e^{\mathbf{i}3\pi/4}, e^{\mathbf{i}5\pi/4}, e^{\mathbf{i}7\pi/4}\}.$$

$3e^{\mathbf{i}\pi/4}, 3e^{\mathbf{i}7\pi/4}$ is the conjugate pair $\frac{3\sqrt{2}}{2} \pm \mathbf{i}\frac{3\sqrt{2}}{2}$.
$3e^{\mathbf{i}3\pi/4}, 3e^{\mathbf{i}5\pi/4}$ is the conjugate pair $\frac{-3\sqrt{2}}{2} \pm \mathbf{i}\frac{3\sqrt{2}}{2}$.
So

$$\begin{aligned} y_h = {} & C_1 + C_2 t + \\ & C_3 e^{\frac{3\sqrt{2}t}{2}} \cos(\frac{3\sqrt{2}t}{2}) + C_4 e^{\frac{3\sqrt{2}t}{2}} \sin(\frac{3\sqrt{2}t}{2}) \\ & + C_5 e^{-\frac{3\sqrt{2}t}{2}} \cos(\frac{3\sqrt{2}t}{2}) + C_6 e^{-\frac{3\sqrt{2}t}{2}} \sin(\frac{3\sqrt{2}t}{2}). \end{aligned}$$

Undetermined Coefficients for Higher Order Equations, Section 4.3

Example E:

$$y^{(11)} + 32y^{(7)} + 256y^{(3)} =$$
$$3t^2 - 7te^{\sqrt{2}t} + e^{-\sqrt{2}t}\sin(\sqrt{2}t) + t^2e^{\sqrt{2}t}\cos(\sqrt{2}t).$$

$$P(r) = r^{11} + 32r^7 + 256r^3 = r^3(r^4 + 16)^2$$

with roots
$0, 0, 0, \sqrt{2} \pm \sqrt{2}\mathbf{i}, \sqrt{2} \pm \sqrt{2}\mathbf{i}, -\sqrt{2} \pm \sqrt{2}\mathbf{i}, -\sqrt{2} \pm \sqrt{2}\mathbf{i}.$

$$\begin{aligned} y_h = {} & C_1 + C_2t + C_3t^2 \\ & C_4e^{\sqrt{2}t}\cos(\sqrt{2}t) + C_5e^{\sqrt{2}t}\sin(\sqrt{2}t) + \\ & C_6te^{\sqrt{2}t}\cos(\sqrt{2}t) + C_7te^{\sqrt{2}t}\sin(\sqrt{2}t) \\ & C_8e^{-\sqrt{2}t}\cos(\sqrt{2}t) + C_9e^{-\sqrt{2}t}\sin(\sqrt{2}t) + \\ & C_{10}te^{-\sqrt{2}t}\cos(\sqrt{2}t) + C_{11}te^{\sqrt{2}t}\sin(\sqrt{2}t). \end{aligned}$$

The associated roots for the four terms of the forcing function are $0, \sqrt{2}, -\sqrt{2} \pm \sqrt{2}\mathbf{i}, \sqrt{2} \pm \sqrt{2}\mathbf{i}$.

The first version of the test function is

$$
\begin{aligned}
Y_p^1 = [At^2 + Bt + C] + [(Dt + E)e^{\sqrt{2}t}] + \\
[Fe^{-\sqrt{2}t}\cos(\sqrt{2}t) + Ge^{-\sqrt{2}t}\sin(\sqrt{2}t)] \\
+[(Ht^2 + It + J)e^{\sqrt{2}t}\cos(\sqrt{2}t) + (Kt^2 + Lt + M)e^{\sqrt{2}t}\sin(\sqrt{2}t)].
\end{aligned}
$$

From the interference with the roots of the homogeneous equation, we must use as the correct test function:

$$
\begin{aligned}
Y_p = t^3[At^2 + Bt + C] + [(Dt + E)e^{\sqrt{2}t}] + \\
t^2[Fe^{-\sqrt{2}t}\cos(\sqrt{2}t) + Ge^{-\sqrt{2}t}\sin(\sqrt{2}t)] \\
+t^2[(Ht^2 + It + J)e^{\sqrt{2}t}\cos(\sqrt{2}t) + (Kt^2 + Lt + M)e^{\sqrt{2}t}\sin(\sqrt{2}t)].
\end{aligned}
$$

Variation of Parameters, Section 3.6

For a general second order linear equation $\mathcal{L}(y) = y'' + py' + qy = f$, suppose we have two independent solutions y_1, y_2 for the homogeneous equation $\mathcal{L}(y) = 0$. So the general solution of the homogeneous equation is $y_h = C_1 y_1 + C_2 y_2$.

We need a particular solution for the equation with forcing term f. We look for $y_p = u_1 y_1 + u_2 y_2$ with u_1, u_2 nonconstant functions to be determined.

We impose two conditions. First, we want y_p to be a solution of $\mathcal{L}(y) = f$.

Second, when we take the derivative $y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$. Our second condition is $u_1' y_1 + u_2' y_2 = 0$.

So when we substitute we have

$$\begin{array}{rrcl} q\times & y_p & = & u_1y_1 + u_2y_2 \\ p\times & y_p' & = & u_1y_1' + u_2y_2' \\ 1\times & y_p'' & = & u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' \\ \hline & f & = & 0 \quad + \quad 0 + u_1'y_1' + u_2'y_2'. \end{array}$$

So u_1', u_2' are determined by two linear equations

$$\begin{array}{rcl} u_1'y_1 + u_2'y_2 & = & 0 \\ u_1'y_1' + u_2'y_2' & = & f. \end{array}$$

Notice that the determinant of coefficients is the Wronskian and so is nonzero.

Example 3.6/ BD6; BDM5 : $y'' + 9y = 9\sec^2(3t)$. The roots of the homogeneous are $\pm 3\mathbf{i}$ and so $y_1 = \cos(3t), y_2 = \sin(3t)$. We look for $y_p = u_1 \cos(3t) + u_2 \sin(3t)$. We have the linear equations:

$$\begin{array}{rcrcl} u_1'(\cos(3t)) & + & u_2'(\sin(3t)) & = & 0 \\ u_1'(-3\sin(3t)) & + & u_2'(3\cos(3t)) & = & 9\sec^2(3t). \end{array}$$

The Wronskian is $3(\cos^2(3t) + \sin^2(3t)) = 3$ and so by Cramer's Rule

$$u_1' = -3\sin(3t)\sec^2(3t) = -3\sec(3t)\tan(3t) \;\Rightarrow\; u_1 = -\sec(3t)$$
$$u_2' = 3\cos(3t)\sec^2(3t) = 3\sec(3t) \;\Rightarrow\; u_2 = \ln|\sec(3t) + \tan(3t)|$$

$y_p = -\sec(3t)\cos(3t) + (\ln|\sec(3t) + \tan(3t)|)\sin(3t) =$
$-1 + (\ln|\sec(3t) + \tan(3t)|)\sin(3t)$.

Example 3.6/ BD7; BDM6 : $y'' + 4y' + 4y = t^{-2}e^{-2t}$. The characteristic equation is $r^2 + 4r + 4 = (r+2)^2 = 0$ and so $y_h = C_1e^{-2t} + C_2te^{-2t}$. We look for $y_p = u_1e^{-2t} + u_2te^{-2t}$. We have the linear equations:

$$\begin{aligned} u_1'e^{-2t} + u_2'te^{-2t} &= 0 \\ u_1'(-2e^{-2t}) + u_2'(-2te^{-2t} + e^{-2t}) &= t^{-2}e^{-2t}. \end{aligned}$$

The Wronskian is e^{-4t} and so

$$\begin{aligned} u_1' = -t^{-1}e^{-4t}/e^{-4t} = -t^{-1} \;\Rightarrow\; & u_1 = -\ln(t) \\ u_2' = t^{-2}e^{-4t}/e^{-4t} = t^{-2} \;\Rightarrow\; & u_2 = -t^{-1}. \end{aligned}$$

$y_g = C_1e^{-2t} + C_2te^{-2t} - \ln(t)e^{-2t} - t^{-1}te^{-2t}$ or

$$y_g = C_1e^{-2t} + C_2te^{-2t} - \ln(t)e^{-2t}.$$

Example 3.6/ BD16 : $(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}$ with $y_1 = e^t, y_2 = t$. Check that y_1 and y_2 are solutions of the homogeneous equation.

Divide by $(1-t)$ to get $r = -2(t-1)e^{-t}$ and so

$$\begin{aligned} u_1' e^t + u_2' t &= 0 \\ u_1' e^t + u_2' &= -2(t-1)e^{-t}. \end{aligned}$$

The Wronskian is $(1-t)e^t$ and so

$$\begin{aligned} u_1' = 2(t-1)te^{-t}/(1-t)e^t = -2te^{-2t} &\Rightarrow \quad u_1 = te^{-2t} + \frac{1}{2}e^{-2t} \\ u_2' = -2(t-1)/(1-t)e^t = 2e^{-t} &\Rightarrow \quad u_2 = -2e^{-t}. \end{aligned}$$

$y_p = (t + \frac{1}{2})e^{-t} - 2te^{-t} = (\frac{1}{2} - t)e^{-t}$.

Spring Problems, Sections 3.7, 3.8

The *restoring force* of a spring is in the opposite direction to its displacement y from equilibrium. In the case of a *linear spring*, which is all we is consider, it is proportional to the displacement and so is $-ky$ with k a positive constant called the *spring constant*.

If there is friction - imagine the spring moving through a liquid like molasses - then the friction force is in the opposite direction to its velocity $v = \frac{dy}{dt} = y'$. We will assume it is proportional to the velocity and so is $-cv$ where c is a positive constant (or zero if there is no friction).

Finally, there may be an external force, called the *forcing term*, $f(t)$ which may not be a constant. It is usually assume periodic.

Thus, the total force is $-ky - cy' + f(t)$. Newton's Third Law says that the total force is proportional to the the acceleration $a = \frac{d^2y}{dt^2} = y''$ and the proportionality constant is the *mass* m. That is, $F = ma$. So the differential equation for the spring is:

$$my'' + cy' + ky = 0.$$

The *weight* w of an object is the force due to gravity. The acceleration due to gravity is a constant (near the surface of the earth) denoted $-g$. (Why the minus sign?) In English units $g = 32ft/sec^2$. Thus, weight and mass are related by $w = mg$.

When a weight is attached to a hanging spring it stretches it a distance denoted ΔL. The equilibrium position is when the spring force exactly balances the weight. That is, $w = k\Delta L$. So the equations which relate the constants in these problems are

$$w = mg, \qquad w = k\Delta L.$$

Suppose there is no friction and no external force so that $my'' + ky = 0$. This is a second order, linear, homogeneous equation with constant coefficients. The characteristic equation is $mr^2 + k = 0$ with roots $\pm\omega\mathbf{i}$ where $\omega = \sqrt{k/m} = \sqrt{g/\Delta L}$, the *natural frequency* of the spring. Thus, the general solution is

$$y = C_1\cos(\omega t) + C_2\sin(\omega t),$$

with the constants $C_1 = y(0)$, the initial position, and $\omega C_2 = y'(0)$, the initial velocity.

Regarding the pair (C_1, C_2) as a point in the plane we can write it as $(A\cos(\phi), A\sin(\phi))$, with $A^2 = C_1^2 + C_2^2$. Remember that $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$ and so

$$y = A\cos(\omega t)\cos(\phi) + A\sin(\omega t)\sin(\phi) = A\cos(\omega t - \phi).$$

If there is damping, and so $c > 0$, then the characteristic equation is $r^2 + 2(c/2m)r + (k/m) = 0$
If the damping is small, with $(c/2m) < \sqrt{k/m}$ then the roots are a complex conjugate pair $-(c/2m) \pm \sqrt{(k/m) - (c/2m)^2}\mathbf{i}$ and the solution is

$$y = e^{-(c/2m)t}[C_1 \cos(\sqrt{(k/m) - (c/2m)^2}\ t) \quad + \\ C_2 \sin(\sqrt{(k/m) - (c/2m)^2}\ t)].$$

If the damping is large, with $(c/2m) > \sqrt{k/m}$ then the roots are real with

$$y = C_1 e^{[-(c/2m) + \sqrt{(c/2m)^2 - (k/m)}]\ t} + C_2 e^{[-(c/2m) - \sqrt{(c/2m)^2 - (k/m)}]\ t}.$$

Finally, if $(c/2m) = \sqrt{k/m}$ then $-(c/2m)$ is a repeated root and so $y = e^{-(c/2m)t}[C_1 + C_2 t]$.
In any case, the solution decays exponentially.

In the undamped case, suppose there is periodic forcing:
$y'' + \omega_0^2 y = \cos(\omega t)$.

The roots of the characteristic equation $r^2 + \omega_0^2 = 0$ are $\pm\omega_0\mathbf{i}$.
ω_0 is the *natural frequency* of the spring.

The associated roots for the forcing term are $\pm\omega\mathbf{i}$.

If $\omega \neq \omega_0$ then the test function $Y_p = A\cos(\omega t) + B\sin(\omega t)$.

If $\omega = \omega_0$ then the test function
$Y_p = At\cos(\omega t) + Bt\sin(\omega t)$. Forcing at the natural frequency of the spring is called *resonance*.

Example 3.8/ BD11; BDM7 : A spring is stretched $6in$ by a mass that weighs $4lb$. The mass is attached to a dashpot with a damping constant of $0.25 lb \cdot sec/ft$ and is acted upon by an external force of $4\cos(2t) lb$.

- Determine the general solution and the steady state response of the system.
- If the given mass is replaced by one of mass m, determine m so that the amplitude of the steady state response is a maximum.

$\Delta L = \frac{1}{2}, w = 4, c = \frac{1}{4}, g = 32$, and so
$k = w/\Delta L = 8, m = w/g = \frac{1}{8}$. So

$$\frac{1}{8}y'' + \frac{1}{4}y' + 8y = 4\cos(2t), \quad \text{or} \quad y'' + 2y' + 64y = 32\cos(2t).$$

$$y_h = e^{-t}[C_1 \cos(\sqrt{63}\ t) \ + \ C_2 \sin(\sqrt{63}\ t)]$$

The particular solution $y_p = A\cos(2t) + B\sin(2t)$ is the steady-state solution.

Substitute into the equation $y'' + 2y' + 64y = 32\cos(2t)$

$$\begin{array}{rcrcr} 64 \times \; [\; y_p & = & A\cos(2t) & + & B\sin(2t)\;] \\ 2 \times \; [\; y_p' & = & 2B\cos(2t) & - & 2A\sin(2t)\;] \\ 1 \times \; [\; y_p'' & = & -4A\cos(2t) & - & 4B\sin(2t)\;] \\ \hline 32\cos 2t & = & (60A + 4B)\cos(2t) & + & (-4A + 60B)\sin(2t). \end{array}$$

$$\begin{array}{rcrcl} 15A & + & B & = & 8 \\ -A & + & 15B & = & 0 \end{array} \quad \Rightarrow \quad (A, B) = \frac{4}{113}(15, 1).$$

The amplitude is $\sqrt{A^2 + B^2} = \frac{4\sqrt{226}}{113}$.

If the mass m is substituted then the equation is $my'' + \frac{1}{4}y' + 8y = 4\cos(2t)$. As there is damping the steady-state solution is again $y_p = A\cos(2t) + B\sin(2t)$.

$$\begin{array}{rcrcr}
8 \times \ [\ y_p & = & A\cos(2t) & + & B\sin(2t)\ ] \\
\dfrac{1}{4} \times \ [y_p' & = & 2Bt\cos(2t) & - & 2At\sin(2t)\ ] \\
\underline{m \times \ [\ y_p''} & \underline{=} & \underline{-4A\cos(2t)} & \underline{-} & \underline{4B\sin(2t)\ ]}
\end{array}$$

$$\begin{aligned}
4\cos 2t \ &= \ ((-4m+8)A + \frac{1}{2}B)\cos(2t) \\
&\quad + \ (-\frac{1}{2}A + (-4m+8)B)\sin(2t).
\end{aligned}$$

$$\begin{array}{rcccl} (-8m+16)A & + & B & = & 8 \\ -A & + & (-8m+16)B & = & 0 \end{array}.$$

The coefficient determinant is $64(2-m)^2+1$ and so

$$(A,B) = \frac{8}{64(2-m)^2+1}(8(2-m),1).$$

The amplitude is $\frac{8}{\sqrt{64(2-m)^2+1}}$ which has its maximum at $m = 2$.

Series Solutions Centered at an Ordinary Point

We are going to look at second order, linear, homogeneous equations of the form $P(x)y'' + Q(x)y' + R(x)y = 0$ with the coefficients P, Q, R polynomials in x. The center $x = a$ is an *ordinary point* when $P(a)$ is not zero so that Q/P and R/P are defined at $x = a$.

We will look for a series solution centered at $x = 0$. Thus,

$$
\begin{aligned}
y &= \Sigma\, a_n x^n. \\
y' &= \Sigma\, n a_n x^{n-1}. \\
y'' &= \Sigma\, n(n-1) a_n x^{n-2}.
\end{aligned}
$$

We use three steps

Step 1: Write a separate series for each term of the polynomials P, Q and R.

Step 2: Shift indices so that each series is represented as powers of x^k.

Step 3: Collect terms to obtain the *recursion formula* for the a_k's with $a_0 = y(0)$ and $a_1 = y'(0)$ the arbitrary constants.

Example: $(2+3x^2)y'' + (x-5x^3)y' + (1-x^2+x^4)y = 0$. So there will be seven series.

$$
\begin{aligned}
2y'' &= \Sigma\, 2n(n-1)a_n x^{n-2} [k=n-2] = \Sigma 2(k+2)(k+1)a_{k+2}x^k.\\
3x^2y'' &= \Sigma\, 3n(n-1)a_n x^n \; [k=n] = \Sigma\, 3k(k-1)a_k x^k.\\
xy' &= \Sigma\, na_n x^n \quad [k=n] = \Sigma\, ka_k x^k.\\
-5x^3y' &= \Sigma - 5na_n x^{n+2} [k=n+2] = \Sigma - 5(k-2)a_{k-2}x^k.\\
1y &= \Sigma\, a_n x^n \quad [k=n] = \Sigma\, a_k x^k.\\
-x^2y &= \Sigma - a_n x^{n+2} [k=n+2] = \Sigma\, - a_{k-2}x^k.\\
x^4y &= \Sigma\, a_n x^{n+4} \quad [k=n+4] = \Sigma\, a_{k-4}x^k.
\end{aligned}
$$

We sum and collect the coefficients of x^k to get the recursion formula.

Recursion Formula:

$$a_{k+2} = \frac{1}{2(k+2)(k+1)}[-3k(k-1)a_k$$
$$-ka_k + 5(k-2)a_{k-2} - a_k + a_{k-2} - a_{k-4}] =$$
$$\frac{1}{2(k+2)(k+1)}[(-k(3k-2)-1)a_k + (5k-9)a_{k-2} - a_{k-4}].$$

Substitute:

$k = 0, \quad a_2 = \frac{1}{4}[-a_0] = -\frac{1}{4}a_0.$

$k = 1, \quad a_3 = \frac{1}{12}[-2a_1] = -\frac{1}{6}a_1.$

$k = 2, \quad a_4 = \frac{1}{24}[-9a_2 + 1a_0] = \frac{1}{24}[\frac{13}{4}a_0] = \frac{13}{96}a_0.$

$k = 3, \quad a_5 = \frac{1}{40}[-22a_3 + 6a_1] = \frac{1}{40}[\frac{29}{3}a_1] = \frac{29}{120}a_1.$

$k = 4, \quad a_6 = \frac{1}{60}[-41a_4 + 11a_2 - a_0] =$
$\frac{1}{60}[-\frac{41 \cdot 13}{96}a_0 - \frac{11}{4}a_0 - a_0] = -\frac{893}{5760}a_0.$

Example: $(1-x)y'' + (x-2x^2)y' + (1-x+x^2)y = 0. y(0) = -12, y'(0) = 12$ So there will again be seven series. Step 1: and Step 2:

$$
\begin{aligned}
1y'' &= \Sigma\ 1n(n-1)a_n x^{n-2}[k=n-2] = \Sigma 1(k+2)(k+1)a_{k+2}x \\
-xy'' &= \Sigma\ -n(n-1)a_n x^{n-1}\ [k=n-1] = \Sigma\ -(k+1)(k)a_{k+1}x^k. \\
xy' &= \Sigma\ na_n x^n \quad [k=n] = \Sigma\ ka_k x^k. \\
-2x^2y' &= \Sigma - 2na_n x^{n+1}[k=n+1] = \Sigma - 2(k-1)a_{k-1}x^k. \\
1y &= \Sigma\ a_n x^n \quad [k=n] = \Sigma\ a_k x^k. \\
-xy &= \Sigma - a_n x^{n+1}[k=n+1] = \Sigma\ -a_{k-1}x^k. \\
x^2y &= \Sigma\ a_n x^{n+2} \quad [k=n+2] = \Sigma\ a_{k-2}x^k.
\end{aligned}
$$

We sum and collect the coefficients of x^k to get the recursion formula.

Recursion Formula:

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[(k+1)(k)a_{k+1}$$

$$-ka_k + 2(k-1)a_{k-1} - a_k + a_{k-1} - a_{k-2}] =$$

$$\frac{1}{(k+2)(k+1)}[((k+1)(k)a_{k+1} - (k+1)a_k + (2k-1)a_{k-1} - a_{k-2}].$$

Substitute: $a_0 = -12, a_1 = 12$

$k = 0, \quad a_2 = \frac{1}{2}[0 - a_0 + 0 + 0] = 6.$

$k = 1, \quad a_3 = \frac{1}{6}[2a_2 - 2a_1 + a_0 + 0] = -4.$

$k = 2, \quad a_4 = \frac{1}{12}[6a_3 - 3a_2 + 3a_1 - a_0] = \frac{1}{2}.$

$k = 3, \quad a_5 = \frac{1}{20}[12a_4 - 4a_3 + 5a_2 - a_1] = 2.$

$k = 4, \quad a_6 = \frac{1}{30}[20a_5 - 5a_4 + 7a_3 - a_2] = \frac{7}{60}$

$$y = -12 + 12x + 6x^2 - 4x^3 + \frac{1}{2}x^4 + 2x^5 + \frac{7}{60}x^6 + \dots$$

Example: $y'' + [2 + (x-1)^2]y' + (x^2 - 1)y = 0$ centered at $x = 1$ So we want a solution of the form $y = \Sigma a_n(x-1)^n$. Let $X = x - 1$ so that $x = X + 1$. Then $x^2 - 1 = X^2 + 2X$ and $y = \Sigma a_n X^n$ with $y'' + [2 + X^2]y' + (X^2 + 2X)y = 0$. Notice that $\frac{dy}{dX} = \frac{dy}{dx}$ because $\frac{dX}{dx} = 1$.

$$
\begin{aligned}
y'' &= \Sigma\, n(n-1)a_n X^{n-2} [k = n-2] = \Sigma (k+2)(k+1)a_{k+2}X^k. \\
2y' &= \Sigma\, 2na_n X^{n-1} \quad [k = n-1] = \Sigma\, 2(k+1)a_{k+1}X^k. \\
X^2y' &= \Sigma\, na_n X^{n+1} \quad [k = n+1] = \Sigma\, (k-1)a_{k-1}X^k. \\
X^2y &= \Sigma\, a_n X^{n+2} \quad [k = n+2] = \Sigma\, a_{k-2}X^k. \\
2Xy &= \Sigma\, 2a_n X^{n+1} \quad [k = n+1] = \Sigma\, 2a_{k-1}X^k.
\end{aligned}
$$

Recursion Formula:

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}].$$

From the recursion formula $a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}]$, we substitute to get $k = 0, \quad a_2 = \frac{1}{2}[-2a_1 - 0 - 0] = -a_1$.

$$
\begin{aligned}
k &= 1, \quad a_3 = \tfrac{1}{6}[-4a_2 - 2a_0 - 0] = \tfrac{2}{3}a_1 - \tfrac{1}{3}a_0. \\
k &= 2, \quad a_4 = \tfrac{1}{12}[-6a_3 - 3a_1 - a_0] = -\tfrac{7}{12}a_1 + \tfrac{1}{12}a_0 \\
k &= 3, \quad a_5 = \tfrac{1}{20}[-8a_4 - 4a_2 - a_1] = \tfrac{23}{60}a_1 - \tfrac{1}{30}a_0. \\
k &= 4, \quad a_6 = \tfrac{1}{30}[-10a_5 - 5a_3 - a_2] = -\tfrac{37}{180}a_1 + \tfrac{1}{15}a_0.
\end{aligned}
$$

$a_0 = y(1)$ and $a_1 = y'(1)$.

If the initial conditions $y(1) = 18, y'(1) = -24$ are given, it is best to substitute immediately. From the recursion formula $a_{k+2} = \frac{1}{(k+2)(k+1)}[-2(k+1)a_{k+1} - (k+1)a_{k-1} - a_{k-2}]$, we substitute to get

$$
\begin{aligned}
k &= 0, \quad a_2 = \tfrac{1}{2}[48 - 0 - 0] = 24. \\
k &= 1, \quad a_3 = \tfrac{1}{6}[-96 - 36 - 0] = -22. \\
k &= 2, \quad a_4 = \tfrac{1}{12}[132 + 72 - 18] = \tfrac{31}{2}. \\
k &= 3, \quad a_5 = \tfrac{1}{20}[-124 - 96 + 24] = -\tfrac{49}{5}. \\
k &= 4, \quad a_6 = \tfrac{1}{30}[98 + 110 - 24] = \tfrac{92}{15}.
\end{aligned}
$$

So that

$$
y = 18 - 24(x-1) + 24(x-1)^2 - 22(x-1)^3 + \frac{31}{2}(x-1)^4 - \frac{49}{5}(x-1)^5 + \frac{92}{15}(x-1)^6 + \dots .
$$