MATH 39100 K Test 1 - Solutions Fall, 2025

(1) (12 points each)Compute the general solution of each of the following differential equations:

(a) $$y' + y^2 \sin(x) = 0.$$

Variables Separable: $\int y^{-2} dy = -\int \sin(x) dx$.

$$-y^{-1} = C + \cos(x) \quad \text{or} \quad y = -\frac{1}{\cos(x) + C}.$$

(b) $$(1+t^2)\frac{dy}{dt} + 4ty = (1+t^2)^{-2}$$

Linear: $\frac{dy}{dt} + (\frac{4t}{1+t^2})y = (1+t^2)^{-3}$

$$\mu = exp(\int \frac{4t}{1+t^2} dt) = exp(2\ln(1+t^2)) = (1+t^2)^2.$$

$$(1+t^2)^2 \frac{dy}{dt} + (4t(1+t^2))y = [(1+t^2)^2 y]' = (1+t^2)^{-1}.$$

So $(1+t^2)^2 y = \arctan(t) + C$ or $y = (\arctan(t) + C)/(1+t^2)^2$.

(c) $y'' - 5y = 0$ (y is a function of x).

Second Order Linear: Characteristic Equation $r^2 - 5 = 0$ with roots $r = \pm\sqrt{5}$.

$$y = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x}.$$

(d) $$ydy \;+\; (6x - 5y)dx \quad = \quad 0.$$

Homogeneous with $\frac{dy}{dx} = \frac{5y-6x}{y}$. Let $z = y/x$.

$$x\frac{dz}{dx} = \frac{5z-6}{z} - z = -\frac{z^2 - 5z + 6}{z}.$$

$$-\int \frac{z}{(z-3)(z-2)}dz = \int -\frac{dx}{x}.$$

$$\frac{z}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z-2}.$$

So $z = A(z-2) + B(z-3)$.
With $z = 3, A = 3$ and with $z = 2, B = -2$.

$$C - \ln(x) = 3\ln(\frac{y-3x}{x}) - 2\ln(\frac{y-2x}{x}).$$

$$3\ln(y-3x) - 2\ln(y-2x) = C, \quad \text{or} \quad \frac{(y-3x)^3}{(y-2x)^2} = C.$$

(2) (12 points each) Solve the following initial value problems:

(a)(a) $\quad y'y'' = 2, \qquad$ with $\quad y(0) = 1, y'(0) = 2.$

Reduction of Order: Let $y' = v$ so that $y'' = \frac{dv}{dt}$.

$$v\frac{dv}{dt} = 2.$$

$$v^2/2 = \int vdv = \int 2dt = 2t + C_1.$$

When $t = 0$, $v = 2$ and so $C_1 = 2$, $v^2 = 4(t+1)$.
$\frac{dy}{dt} = v = 2\sqrt{t+1}$ (positive square root because $v = 2$ when $t = 0$.So

$$y = 2\int (1+t)^{1/2}dt = \frac{4}{3}(t+1)^{3/2} + C_2.$$

When $t = 0$, $y = 1$ and so $C_2 = -\frac{1}{3}$.

$$y = \frac{4(t+1)^{3/2} - 1}{3}.$$

(b) $\quad (1 + e^{xy}(y \cos x - \sin x))\ dx + (xe^{xy} \cos x - y)\ dy \quad = \quad 0$
with $y(0) = 2\pi$.

$$\frac{\partial}{\partial y}(1+e^{xy}(y\cos x - \sin x)) = e^{xy}(xy\cos x - x\sin x + \cos x) = \frac{\partial}{\partial x}(xe^{xy}\cos x - y)$$

Exact Equation: $F = \int (xe^{xy} \cos x - y)\ dy = e^{xy} \cos x - \frac{y^2}{2} + H(x)$.

$$\frac{\partial F}{\partial x} = e^{xy}(y\cos x - \sin x) + H'(x) = 1 + e^{xy}(y\cos x - \sin x).$$

$H'(x) = 1$ and so $H(x) = x$.
$e^{xy} \cos x - \frac{y^2}{2} + x = C$. Since $y(0) = 2\pi$, $1 - 2\pi^2 + 0 = C$ and so $C = 1 - 2\pi^2$.

(3)(9 points) Assume that y_1, y_2 are solutions of the equation $y'' + py' + qy = 0$ where p and q are functions of t.

(a) Define the *Wronskian* W of y_1, y_2. $W = y_1 y_2' - y_2 y_1'$.

(b) Prove Abel's Theorem. That is, show that the Wronskian satisfies a first order differential equation.

$0 = y_1 y_2 - y_2 y_1$.
$W = y_1 y_2' - y_2 y_1'$.
$W' = y_1 y_2" - y_2 y_1"$.
Multiply the first row by q, the second row by p, the third row by 1 and add.
Because y_1 and y_2 are solutions we obtain:
$0 + pW + W' \ = \ 0 + 0$. That is, $\frac{dW}{dt} + pW = 0$.

(4) (4) (10 points)A 400 gallon tank contains 120 gallons of water in which is dissolved 20 pounds of salt. Starting at time t = 0, a solution with a concentration of 5 pounds per gallon is pumped into the tank at a rate of 7 gallons per minute. At the same time, the well-stirred mixture is pumped out at the rate of 5 gallons per minute.

Set up an initial value problem (differential equation and initial conditions) for the amount $Q(t)$ of salt (in pounds) in the tank at time t until the tank is full. You need not solve the equation.

$\frac{dV}{dt} = 7 - 5$ in gal/min with $V_0 = 120$. So $V = 120 + 2t$.
In $pounds/min$

$$\frac{dQ}{dt} = 7 \cdot 5 - 5\frac{Q}{V} = 35 - \frac{5}{120 + 2t}Q$$

with $Q_0 = 20$.

(5) (9 points) I borrow $\$15,000$ at an interest rate of 6% per year, compounded continuously. I pay off the loan continuously at a rate of $1200 per year. Set up an initial value problem (differential equation and initial conditions) whose solution is the quantity $S(t)$ of dollars that I owe at time t, until the loan is paid off. You need not solve the equation.

$$\frac{dP}{dt} = .06P - 1200,$$

with $P_0 = 15,000$.

Remember to show your work. Good luck.