(1) (12 points each)Compute the general solution of each of the following differential equations:

(a) $\frac{dy}{dx} = \frac{y+x^2y}{x}$.

Variables Separable: $\int \frac{dy}{y} = \int \frac{1+x^2y}{x}dx$.
So $\ln y = \ln x + \frac{x^2}{2} + C$, or $y = Cxexp(\frac{x^2}{2})$

(b) $(1+t^2)\frac{dy}{dt} + 2ty = \frac{4}{t}$

Linear: Divide by $1+t^2$. $\mu = exp(\int \frac{2t}{1+t^2}dt) = (1+t^2)$
$(1+t^2)\frac{dy}{dt} + 2ty = [(1+t^2)y]' = \frac{4}{t}$.
So $(1+t^2)y = 4\ln(t) + C$.
Exact: $(1+t^2)dy + (2ty - \frac{4}{t})dt = 0$.
$F(t,y) = (1+t^2)y + H(t)$. So $2ty + H'(t) = \frac{\partial F}{\partial t} = 2ty - \frac{4}{t}$.
$H(t) = -4\ln(t)$. $(1+t^2)y - 4\ln(t) = C$.

(c) $\frac{dy}{dx} = \frac{y+x}{y^2-x}$.

Not Separable, Linear or Homogeneous. Rewrite $(y^2-x)dy - (x+y)dx = 0$.
Exact: $\frac{\partial(y^2-x)}{\partial x} = -1 = \frac{\partial(-x-y)}{\partial y}$.
$F(x,y) = \int(y^2-x)dy = \frac{y^3}{3} - xy + H(x)$.
So $\frac{\partial F}{\partial x} = -y + H'(x) = -x - y$. and $H'(x) = -x$.
$F(x,y) = \frac{y^3}{3} - xy - \frac{x^2}{2}$. and the solution is

$$\frac{y^3}{3} - xy - \frac{x^2}{2} = C.$$

(d) $y\ \frac{dy}{dx} = (3x+2y)$.

Homogeneous with $\frac{dy}{dx} = \frac{3x+2y}{y}$. Let $z = y/x$.
$x\frac{dz}{dx} = \frac{3+2z}{z} - z = -\frac{z^2-2z-3}{z}$.
$-\int \frac{z}{(z-3)(z+1)}dz = \int \frac{dx}{x}$.
$\frac{z}{(z-3)(z+1)} = \frac{A}{z-3} + \frac{B}{z+1}$. So $z = A(z+1) + B(z-3)$.
With $z = 3, A = 3/4$ and with $z + -1, B = 1/4$.

$$C + \ln(x) = -\frac{3}{4}\ln(\frac{y-3x}{x}) - \frac{1}{4}\ln(\frac{y+x}{x}).$$

$$3\ln(y-3x) + \ln(y+x) = C, \quad \text{or} \quad (y-3x)(y+x) = C.$$

(2) (12 points each) Solve the following initial value problems:

(a) $\quad y'' + t(y')^2 = 0, \qquad$ with $\quad y(0) = 2, y'(0) = 1.$

Let $y' = v$ so that $y'' = \frac{dv}{dt}$.
$\frac{dv}{dt} = -tv^2$. $-v^{-1} = \int v^{-2} dv = \int -t dt = -t^2/2 + C_1$.
When $t = 0$, $v = 1$ and so $C_1 = -1$.
$\frac{dy}{dt} = v = \frac{2}{2+t^2} = \frac{1}{1+(t/\sqrt{2})^2}$. So $y = \sqrt{2}\arctan(t/\sqrt{2}) + C_2$.
When $t = 0$, $y = 2$ and so $C_2 = 2$. $y = \sqrt{2}\arctan(t/\sqrt{2}) + 2$.

(b) $\quad ((1 + xy)e^{xy} - 1)\ dx + (x^2 e^{xy} + 2y)\ dy \quad = \quad 0 \quad$ with $\quad y(0) = 3.$

$\frac{\partial}{\partial y}(1 + xy)e^{xy} - 1) = (2x + x^2 y)e^{xy} = \frac{\partial}{\partial x}(x^2 e^{xy} + 2y)$
Exact Equation: $F = \int x^2 e^{xy} + 2y\ dy = xe^{xy} + y^2 + H(x)$.
$\frac{\partial F}{\partial x} = (1 + xy)e^{xy} + H'(x) = (1 + xy)e^{xy} - 1$.
$H'(x) = -1$ and so $H(x) = -x$.
$xe^{xy} + y^2 - x = C$. Since $y(0) = 3$, $0 + 4 - 0 = C$ and so $C = 9$.

(3)(9 points) (a) Assume that y_1, y_2 are solutions of the equation $y'' + py' + qy = 1$ where p and q are functions of t. Show that $y_1 + y_2$ is NOT a solution of this equation.

Because y_1 and y_2 are solutions we have
$y_1'' + py_1' + qy_1 = 1$ and $y_2'' + py_2' + qy_2 = 1$. Adding, we get that

$$(y_1 + y_2)'' + p(y_1 + y_2)' + q(y_1 + y_2) = 2 \neq 1.$$

(b) Compute the *Wronskian* W of the pair $y_1 = e^{2x}, y_2 = xe^x$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^x \\ 2e^{2x} & (1+x)e^x \end{vmatrix}$$

$$= (1 + x)e^{3x} - 2xe^{3x} = (1 - x)e^{3x}.$$

(4) (10 points)A 200 gallon tank contains 80 gallons of water in which is dissolved 10 pounds of salt. Starting at time t = 0, a solution with a concentration of 2 pounds per gallon is pumped into the tank at a rate of 5 gallons per minute. At the same time, the well-stirred mixture is pumped out at the rate of 2 gallons per minute.

Set up an initial value problem (differential equation and initial conditions) for the amount $Q(t)$ of salt (in pounds) in the tank at time t until the tank is full. You need not solve the equation.

$\frac{dV}{dt} = 5 - 2$ in gal/min with $V_0 = 80$. So $V = 80 + 3t$.
In $pounds/min$ $\frac{dQ}{dt} = 5 \cdot 2 - 2\frac{Q}{V}$.

$\frac{dQ}{dt} = 10 - \frac{2}{80+3t}Q$ with $Q_0 = 10$.

(5) (9 points) I borrow \$10,000 at an interest rate of 1% per year, compounded continuously. I pay off the loan continuously at a rate of \$750 per year. Set up an initial value problem (differential equation and initial conditions) whose solution is the quantity $S(t)$ of dollars that I owe at time t, until the loan is paid off. You need not solve the equation.

$\frac{dP}{dt} = .01P - 750$, with $P_0 = 10,000$.

Remember to show your work. Good luck.