

Math 39100 K (19392)

- Homework Solutions - Post 01

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Variables Separable

You will need to remember:

$$\frac{d \tan(x)}{dx} = \sec^2(x) \quad \text{and} \quad \frac{d \sec(x)}{dx} = \sec(x) \tan(x).$$

$$\int \tan(x) dx = \ln |\sec(x)| + C \quad \text{and}$$

$$\int \sec(x) dx = \ln |\sec(x) + \tan(x)| + C.$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x).$$

$$\text{So: } \cos^2(x) = \frac{1}{2}[1 + \cos(2x)], \quad \text{and} \quad \sin^2(x) = \frac{1}{2}[1 - \cos(2x)].$$

Also remember:

$$\int \frac{x}{1+x^2} dx = \frac{1}{2} \ln(1+x^2) + C$$

by u -substitution, but

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C.$$

Variables Separable

$$2.2/\text{BD11, BDM11} : xdx + ye^{-x}dy = 0, y(0) = 1.$$

$$\int y dy = -\int xe^x dx.$$

Use Integration by Parts.

$$y^2 = 2e^x - 2xe^x + C.$$

$y = 1$ when $x = 0$ and so

$$1 = 2 - 0 + C \text{ or } -1 = C.$$

$$y = +\sqrt{2e^x - 2xe^x - 1}.$$

Plus sign from initial condition.

Homogeneous

$$2.2/ \text{BD33; BDM28} : x \frac{dz}{dx} + z = \frac{4z-3}{2-z}.$$

$$x \frac{dz}{dx} = \frac{z^2 + 2z - 3}{2 - z} = \frac{(z - 1)(z + 3)}{2 - z}.$$

$$\frac{2 - z}{(z - 1)(z + 3)} = \frac{A}{(z - 1)} + \frac{B}{(z + 3)}.$$

$$2 - z = A(z + 3) + B(z - 1).$$

Substitute $z = -3, 1$ to get

$$B = -\frac{5}{4}, \quad A = \frac{1}{4}.$$

$$\ln |x| + C = +\frac{1}{4} \ln |z - 1| - \frac{5}{4} \ln |z + 3|.$$

$$\ln |x| + C = +\frac{1}{4} \ln |y - x| - \frac{5}{4} \ln |y + 3x| - \frac{1}{4} \ln |x| + \frac{5}{4} \ln |x|.$$

$$C = \frac{|y - x|}{|y + 3x|^5}.$$

Linear Equations

$$2.1/ \text{BD6} : ty' + 2y = \sin(t).$$

Make the coefficient of y' equal to 1:

$$\frac{dy}{dt} + \left(\frac{2}{t}\right)y = \frac{\sin(t)}{t}.$$

$$\mu = \exp\left[\int \frac{2}{t} dt\right] = \exp[2 \ln(t)] = t^2.$$

$$t^2 \frac{dy}{dt} + 2ty = [t^2 y]' = t \sin(t).$$

$$t^2 y = C - t \cos(t) + \sin(t).$$

Exact Equations

$$2.6/ \text{BD9; BDM6} : Mdx + Ndy = (ye^{xy} \cos(2x) - 2e^{xy} \sin(2x) + 2x)dx + (xe^{xy} \cos(2x) - 3)dy.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^{xy} \cos(2x) + xye^{xy} \cos(2x) - 2xe^{xy} \sin(2x).$$

$$F = \int (xe^{xy} \cos(2x) - 3) dy = e^{xy} \cos(2x) - 3y + H(x).$$

$$\begin{aligned} \frac{\partial F}{\partial x} &= ye^{xy} \cos(2x) - 2e^{xy} \sin(2x) + H'(x) \\ &= ye^{xy} \cos(2x) - 2e^{xy} \sin(2x) + 2x. \end{aligned}$$

$$H(x) = x^2 \text{ and so } e^{xy} \cos(2x) - 3y + x^2 = C.$$

Tank Problems

2.3/ BD2; BDM2 : $V_0 = 120L$, $S_0 = 0$. Input and Output $2L/min$. Concentration of Input $\gamma g/L$.

In L/min : $\frac{dV}{dt} = 2 - 2 = 0$. So $V = V_0 = 120$.

In g/min : $\frac{dS}{dt} = 2 \cdot \gamma - 2 \frac{S}{V} = \frac{120\gamma - S}{60}$.

$$\frac{dS}{S - 120\gamma} = -\frac{dt}{60}; \quad S - 120\gamma = C \exp[-t/60].$$

$$C = -120\gamma; \quad S = 120\gamma(1 - \exp[-t/60]).$$

As $t \rightarrow \infty$ S tends to 120γ .

2.3/ BD4 : $V_0 = 200\text{gal}$, $S_0 = 100\text{lb}$. Input $3\text{gal}/\text{min}$, Output $2\text{gal}/\text{min}$. Concentration of Input $1\text{lb}/\text{gal}$.

In gal/min : $\frac{dV}{dt} = 3 - 2 = 1$. So $V = V_0 + t = 200 + t$.

In lb/min : $\frac{dS}{dt} = 3 \cdot 1 - 2\frac{S}{V} = 3 - \frac{2S}{200+t}$.

$$\frac{dS}{dt} + \frac{2}{200+t}S = 3; \quad \mu = (200+t)^2.$$

$$[(200+t)^2 S]' = 3(200+t)^2; \quad (200+t)^2 S = (200+t)^3 + C;$$

$$\text{So } C = 200^2 100 - 200^3 = -200^2 100.$$

$$S = (200+t) - 100 \left(\frac{200}{(200+t)} \right)^2; \quad S/V = 1 - (1/2) \left(\frac{200}{(200+t)} \right)^3.$$

Interest Problems

2.3/ BD8; BDM6 : Initial Amount $S_0 = 0$. Interest Rate $r\%/yr$. Input $k\$/yr$.

$$\frac{dS}{dt} = rS + k; \quad \frac{dS}{S + (k/r)} = rdt.$$

$$S + (k/r) = Ce^{rt}; \quad C = (k/r).$$

$$S = (k/r)(e^{rt} - 1)$$

(b) $1,000,000 = (k/.075)(e^{.075 \cdot 40} - 1)$.

(c) $1,000,000 = (4000/r)(e^{40r} - 1)$. Can't solve explicitly.

2.3/ BD9; BDM7 : Initial Amount $S_0 = 8000$. Interest Rate .1%/yr. Output k \$/yr. $S_3 = 0$.

$$\frac{dS}{dt} = .1S - k; \quad \frac{dS}{S - 10k} = .1dt.$$

$$S - 10k = Ce^{.1t}; \quad 8000 - 10k = C.$$

$$S = 8000e^{.1t} - 10k(e^{.1t} - 1); \quad k = \frac{800e^{.3}}{e^{.3} - 1} = 3087.$$

Interest paid = $3k - 8000 = 1261$.

Reduction of Order

$$2.9/ \text{BD45} : 2y^2v \frac{dv}{dy} + 2yv^2 = 1, \quad \text{or}$$

$$(2y^2v)dv + (2yv^2 - 1)dy = 0.$$

$$\text{Exact: } y^2v^2 - y = C_1 \quad \text{and so } \frac{dy}{dx} = v = \pm \frac{\sqrt{C_1+y}}{y}.$$

$$\text{Let } u = C_1 + y$$

$$\pm x + C_2 = \int dx = \int \frac{u - C_1}{\sqrt{u}} du = \frac{2}{3}u^{3/2} - 2C_1u^{1/2}.$$

$$\pm x + C_2 = \sqrt{C_1 + y} \left[\frac{2}{3}(C_1 + y) - 2C_1 \right] = \sqrt{C_1 + y} \left[\frac{2}{3}(y - 2C_1) \right].$$

Miscellaneous Problems

BD1; BDM1 : Linear: Rewrite as: $\frac{dy}{dx} + \frac{2}{x}y = x^2$.

$$\mu = \exp\left[\int \frac{2}{x} dx\right] = \exp[2 \ln(x)] = x^2.$$

$$[x^2 y]' = x^2 \frac{dy}{dx} + 2xy = x^4.$$

$$x^2 y = \frac{x^5}{5} + C; \quad y = \frac{x^3}{5} + Cx^{-2}.$$

BD2; BDM2 : Variables separable: $2y + \cos y = x + \sin x + C$.

BD 3; BDM3 : Not separable, linear, or homogeneous. Exact:
Rewrite as: $(3 + 3y^2 - x)dy - (2x + y)dx = 0$.

$$\frac{\partial}{\partial x}(3 + 3y^2 - x) = -1 = \frac{\partial}{\partial y} - (2x + y).$$

$$F(x, y) = \int (3 + 3y^2 - x)dy = 3y + y^3 - xy + H(x).$$

$$\frac{\partial F}{\partial x} = -y + H'(x) = -(2x + y).$$

$$H'(x) = -2x; \quad H(x) = -x^2.$$

$$F(x, y) = 3y + y^3 - xy - x^2; \quad \text{Solution: } 3y + y^3 - xy - x^2 = C.$$

$$y(0) = 0; \quad \text{and so } C = 0.$$

BD4; BDM4 : Variables Separable:

$$\frac{dy}{dx} = 3(1 - 2x) + y(1 - 2x) = (3 + y)(1 - 2x).$$

$$\int \frac{dy}{3 + y} = \int (1 - 2x) dx; \quad \ln(3 + y) = x - x^2 + C.$$

$$y = -3 + C \exp[x - x^2].$$

Linear: $\frac{dy}{dx} + (2x - 1)y = 3 - 6x.$

$$\mu = \exp[-x + x^2]; \quad \exp[-x + x^2]y = \int (3 - 6x) \exp[-x + x^2] dx.$$

$$u\text{-sub} : u = -x + x^2 : \exp[-x + x^2]y = -3 \exp[-x + x^2] + C.$$

BD5; BDM5 : Exact: $(x^2 + 2xy)dy + (2xy + y^2 + 1)dx = 0$.

$$\frac{\partial}{\partial x}(x^2 + 2xy) = 2x + 2y = \frac{\partial}{\partial y}(2xy + y^2 + 1).$$

$$F(x, y) = \int (x^2 + 2xy)dy = x^2y + xy^2 + H(x).$$

$$\frac{\partial F}{\partial x} = 2xy + y^2 + H'(x) = (2xy + y^2 + 1).$$

$$H'(x) = 1; H(x) = x; F(x, y) = x^2y + xy^2 + x$$

$$x^2y + xy^2 + x = C.$$

BD6; BDM6 : Linear, Combine y terms: $\frac{dy}{dx} + (1 + \frac{1}{x})y = \frac{1}{x}$.

$$\mu = \exp\left[\int \left(1 + \frac{1}{x}\right) dx\right] = xe^x.$$

$$[xe^x y]' = xe^x \frac{dy}{dx} + (x + 1)e^x y = e^x.$$

$$xe^x y = e^x + C; \quad y(1) = 0; \quad 0 = e + C; \quad C = -e.$$

$$y = (1 - e^{(1-x)})/x.$$

BD8; BDM7 : Linear; BD9; BDM8 : Exact

BD10; BDM9 : Variables Separable:

$$(x^2 + x - 1)y + x^2(y - 2)\frac{dy}{dx} = 0.$$

$$\frac{y - 2}{y} dy = -\frac{x^2 + x - 1}{x^2} dx.$$

$$y - 2 \ln |y| = C - x - \ln |x| - x^{-1}.$$

BD11; BDM10 : Exact; BD12 : Linear;

BD14; BDM11 : Exact: $F(x, y) = xy + \frac{x^2}{2} + y^2$.

$$xy + \frac{x^2}{2} + y^2 = C; \quad y(2) = 3, \quad 6 + 2 + 9 = C; \quad xy + \frac{x^2}{2} + y^2 = 17.$$

Homogeneous: $\frac{dy}{dx} = -\frac{x+y}{x+2y}$; Let $z = y/x$.

$$z \frac{dz}{dx} = -z - \frac{1+z}{1+2z} = -\frac{2z^2 + 2z + 1}{1+2z}.$$

$$\int \frac{1+2z}{2z^2 + 2z + 1} dz = -\int \frac{dx}{x}.$$

$$u - \text{sub} : \frac{1}{2} \ln |2z^2 + 2z + 1| = -\ln x + C;$$

$$2(y/x)^2 + 2(y/x) + 1 = Cx^{-2}; \quad 2y^2 + 2xy + x^2 = C; \quad C = 34.$$

$$\text{BD28 : } (2y + 3x) = -x \frac{dy}{dx}.$$

$$\text{Linear: } \frac{dy}{dx} + \frac{2}{x}y = -3.$$

$$\mu = x^2; \quad [x^2y]' = -3x^2; \quad x^2y = -x^3 + C; \quad y = -x + Cx^{-2}.$$

$$\text{Homogeneous: } \frac{dy}{dx} = -2(y/x) - 3; \quad \text{Let } z = y/x.$$

$$\frac{dz}{dx} = -3 - 3z; \quad \frac{dz}{1+z} = -3 \frac{dx}{x}.$$

$$\ln |1+z| = -3 \ln |x| + C; \quad 1 + (y/x) = Cx^{-3}; \quad x^3 + x^2y = C.$$

BD29; BDM22 : Homogeneous (not Exact): $x \frac{dz}{dx} + z = \frac{1+z}{1-z}$.

$$\int \frac{1-z}{1+z^2} dz = \int \frac{dx}{x}.$$

$$\arctan(z) - \frac{1}{2} \ln(1+z^2) = \ln(x) + C = \frac{1}{2} \ln(x^2) + C.$$

$$\arctan(y/x) - \frac{1}{2} \ln(x^2 + y^2) = C.$$

Example $y'' - y = 0$ via Reduction of Order:

$$v \frac{dv}{dy} = y, \quad v^2 = y^2 + C_1, \quad \frac{dy}{dx} = v = \pm \sqrt{y^2 + C_1}.$$

By Trig Substitution

$$\pm x + C_2 = \ln |y + \sqrt{y^2 + C_1}|, \quad \sqrt{y^2 + C_1} = C_2 e^{\pm x} - y.$$

Square to get $C_1 = C_2^2 e^{\pm 2x} - 2C_2 y e^{\pm x}$. Finally,

$$y = C_1 e^x + C_2 e^{-x}.$$

Instead, use $y = e^{rx}$ as test function. Characteristic Equation $r^2 - 1 = 0$ with roots $r = \pm 1$. So directly,

$$y = C_1 e^x + C_2 e^{-x}.$$