(1) (12 points each)Compute the general solution of each of the following differential equations:

(a) $\frac{dy}{dx} = \frac{x}{y+x^2y}$.

Variables Separable: $\int y dy = \int \frac{x dx}{1+x^2}$.
So $y^2 = \ln(1+x^2) + C$.

(b) $t\frac{dy}{dt} + 4y = \frac{4}{t}$

Linear: Divide by t. $\mu = exp(int \frac{4}{t} dt) = t^4$.
$t^4 \frac{dy}{dt} + 4t^3 y = [t^4 y]' = 4t^2$.
So $t^4 y = \frac{4}{3}t^3 + C$.

(c) $y'' - 3y' = 0$ (y is a function of x).

Second Order Linear: Characteristic Equation $r^2 - 3r = 0$ with roots $r = 0, 3$.
$y = C_1 + C_2 e^{3x}$.

(d) $xy\ dy = (x^2 + y^2)\ dx$.

Homogeneous with $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$. Let $z = y/x$.
$x\frac{dz}{dx} = \frac{1+z^2}{z} - z = \frac{1}{z}$.
$\int z dz = \int \frac{dx}{x}$.
$z^2/2 = \ln(x) + C$. So $y^2 = 2x^2(\ln(x) + C)$.

(2) (12 points each) Solve the following initial value problems:

(a) $y'' + t(y')^2 = 0$, with $y(0) = 2, y'(0) = 1$.

Let $y' = v$ so that $y'' = \frac{dv}{dt}$.
$\frac{dv}{dt} = -tv^2$. $-v^{-1} = \int v^{-2} dv = \int -t dt = -t^2/2 + C_1$.
When $t = 0$, $v = 1$ and so $C_1 = -1$.
$\frac{dy}{dt} = v = \frac{2}{2+t^2} = \frac{1}{1+(t/\sqrt{2})^2}$. So $y = \sqrt{2}\arctan(t/\sqrt{2}) + C_2$.
When $t = 0$, $y = 2$ and so $C_2 = 2$. $y = \sqrt{2}\arctan(t/\sqrt{2}) + 2$.

(b) $(1 - y\sin(xy))\ dx + (2y - x\sin(xy))\ dy = 0$ with $y(0) = 2$.

$\frac{\partial}{\partial y}(1 - y\sin(xy)) = -\sin(xy) - xy\cos(xy) = \frac{\partial}{\partial x}(2y - x\sin(xy))$
Exact Equation: $F = \int(1 - y\sin(xy))\ dx = x + \cos(xy) + H(y)$.
$\frac{\partial F}{\partial y} = -x\sin(xy) + H'(y) = 2y - x\sin(xy)$.
$H'(y) = 2y$ and so $H(y) = y^2$.
$x + \cos(xy) + y^2 = C$. Since $y(0) = 2$, $0 + 1 + 4 = C$ and so $C = 5$.

(3)(9 points) Assume that y_1, y_2 are solutions of the equation $y''+py'+qy=0$ where p and q are functions of t.

(a) Define the *Wronskian* W of y_1, y_2. $W = y_1y_2' - y_2y_1'$.

(b) Prove Abel's Theorem. That is, show that the Wronskian satisfies a first order differential equation.

$0 = y_1y_2 - y_2y_1$.
$W = y_1y_2' - y_2y_1'$.
$W' = y_1y_2" - y_2y_1"$.
Multiply the first row by q, the second row by p, the third row by 1 and add.
Because y_1 and y_2 are solutions we obtain:
$0 + pW + W' = 0 + 0$. That is, $\frac{dW}{dt} + pW = 0$.

(4) (10 points)A 200 gallon tank contains 80 gallons of water in which is dissolved 10 pounds of salt. Starting at time t = 0, a solution with a concentration of 2 pounds per gallon is pumped into the tank at a rate of 5 gallons per minute. At the same time, the well-stirred mixture is pumped out at the rate of 2 gallons per minute.

Set up an initial value problem (differential equation and initial conditions) for the amount $Q(t)$ of salt (in pounds) in the tank at time t until the tank is full. You need not solve the equation.

$\frac{dV}{dt} = 5 - 2$ in gal/min with $V_0 = 80$. So $V = 80 + 3t$.
In $pounds/min$ $\frac{dQ}{dt} = 5 \cdot 2 - 2\frac{Q}{V}$.
$\frac{dQ}{dt} = 10 - \frac{2}{80+3t}Q$ with $Q_0 = 10$.

(5) (9 points) I borrow \$10,000 at an interest rate of 1% per year, compounded continuously. I pay off the loan continuously at a rate of \$750 per year. Set up an initial value problem (differential equation and initial conditions) whose solution is the quantity $S(t)$ of dollars that I owe at time t, until the loan is paid off. You need not solve the equation.

$\frac{dP}{dt} = .01P - 750$, with $P_0 = 10,000$.

Remember to show your work. Good luck.