Math 39100 K (10224) - Homework Solutions - Post 02

Ethan Akin
Email: eakin@ccny.cuny.edu

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Chapter 3, Section 3.3 - BD 10; BDM 7 : $y'' + 2y' + 2y = 0$ with characteristic equation $r^2 + 2r + 2 = 0$.

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Contrast with $y'' + 2y' - 2y = 0$ with characteristic equation $r^2 + 2r + 2 = 0$. Roots: $r = [-2 \pm \sqrt{4+8}]/2 = -1 \pm \sqrt{3}$.

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Contrast with $y'' + 2y' - 2y = 0$ with characteristic equation $r^2 + 2r + 2 = 0$. Roots: $r = [-2 \pm \sqrt{4+8}]/2 = -1 \pm \sqrt{3}$.

$$y = C_1 e^{(-1+\sqrt{3})t} + C_2 e^{(-1-\sqrt{3})t}.$$

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$$\begin{aligned} y &= C_1 \cos(2t) + C_2 \sin(2t), \\ y' &= -2C_1 \sin(2t) + 2C_2 \cos(2t). \end{aligned}$$

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$y(0) = 0$ and $y'(0) = 1$. So $0 = C_1(1) + C_2(0)$ and $1 = -2C_1(0) + 2C_2(1)$..

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$C_1 = 0$, $C_2 = \frac{1}{2}$ and so

$$y = \sin(2t)/2$$

.

Example 3.4/25 : $t^2y'' + 3ty' + y = 0, t > 1$ with $y_1(t) = t^{-1}$.

$$\begin{array}{rcccccccc}
1 \times & y_2 & = & ut^{-1} & & & & & \\
3t \times & y_2' & = & -ut^{-2} & + & u't^{-1} & & & \\
t^2 \times & y_2'' & = & 2ut^{-3} & - & 2u't^{-2} & + & u''t^{-1} \\
& & & & & & & & \\
& 0 & = & 0 & + & u' & + & u''t
\end{array}$$

Let $v = u'$ and $v' = u''$. $t\frac{dv}{dt} = -v$.

$$\ln v = -\ln t, \qquad \text{and so} \quad \frac{du}{dt} = v = t^{-1}.$$

Hence, $u = \int t^{-1}dt = \ln t$.

$$y_2 \;=\; ut^{-1} \;=\; t^{-1}\ln t.$$

Example 3.5/9 : $2y'' + 3y' + y = t^2 + 3\sin t$.

1. Homogeneous Equation $2y'' + 3y' + y = 0$ has characteristic equation $2r^2 + 3r + 1 = (2r + 1)(r + 1) = 0$. So $y_h = C_1 e^{-t/2} + C_2 e^{-t}$.

2. t^2 has associated root 0 and $3\sin t$ has associated root the conjugate pair $\pm\mathbf{i}$. So our first guess for the test solution is $Y = At^2 + Bt + C + D\cos t + E\sin t$. Since none of the associated roots is a root of the characteristic equation, this is what we use for Y_p.

3. Substitute in the equation.

$$\begin{array}{rcllllll}
1 \times Y_p & = & At^2 & + Bt & + C & + D\cos t & + E\sin t, \\
3 \times Y_p' & = & & 6At & + 3B & + 3E\cos t & - 3D\sin t, \\
2 \times Y_p'' & = & & & +4A & - 2D\cos t & - 2E\sin t.
\end{array}$$

So $t^2 + 3\sin t = At^2 + (B + 6A)t + (C + 3B + 4A)$
$+(-D + 3E)\cos t + (-E - 3D)\sin t$.

$A = 1$.
$B + 6A = 0$, and so $B = -6$.
$C + 3B + 4A = 0$, and so $C = 14$.

$-D + 3E = 0$ and so $D = 3E$.
$-E - 3D = 3$ and so $-10E = 3$.
Thus, $E = -3/10, \qquad D = -9/10$.

$$y_g = C_1 e^{-t/2} + C_2 e^{-t} + t^2 - 6t + 14 - (9/10)\cos t - (3/10)\sin t.$$

Example 3.5/17 :
$y'' - 2y' + y = te^t + 4, \quad y(0) = 1, y'(0) = 1.$

1. Homogeneous Equation $y'' - 2y' + y = 0$ has characteristic equation $r^2 - 2r + 1 = (r-1)^2 = 0$. So $y_h = C_1 e^t + C_2 te^t$.

2. te^t has associated root 1 and 4 has associated root 0. So our first guess for the test solution is $Y = Ate^t + Be^t + C$. Since 1 is a root for the homogeneous, we must multiply the block $Ate^t + Be^t$ first by t and then by another t since 1 is a repeated root. So $Y_p = At^3e^t + Bt^2e^t + C$.

3. Substitute in the equation. Leave columns for the te^t and e^t terms.

$$
\begin{array}{rcllll}
1 \times Y_p & = & At^3e^t & + Bt^2e^t & + & +C, \\
-2 \times Y_p' & = & -2At^3e^t & + (-6A - 2B)t^2e^t & - 4Bte^t & , \\
1 \times Y_p'' & = & At^3e^t & + (6A + B)t^2e^t & + (6A + 4B)te^t & + 2Be^t.
\end{array}
$$

So $te^t + 4 = 0 + 0 + 6Ate^t + 2Be^t + C$, Notice that the first two columns on the right add up to 0.
$A = 1/6, B = 0, C = 4$, So that

$$\begin{array}{rcl} y_g & = & C_1 e^t \quad + C_2 te^t \qquad + \quad (1/6)t^3 e^t \quad + 4, \\ y_g' & = & (C_1 + C_2)e^t \;\; + C_2 te^t + \;\; (1/2)t^2 e^t \;\; + (1/6)t^3 e^t. \end{array}$$

$1 = y(0) = C_1 + 4, \qquad 1 = y'(0) = C_1 + C_2.$

$C_1 = -3, \quad C_2 = 4.$

$$y \;=\; -3e^t \;+\; 4te^t \;+\; (1/6)t^3 e^t \;+\; 4.$$

For 19-23, just solve the homogeneous equation and get the test function $Y(t)$. Don't try to solve for the coefficients. Example $3.5/21$: $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$. The characteristic equation of the homogeneous equation is $r^2 + 3r = 0$ with roots $r = 0, -3$. So $y_h = C_1 + C_2e^{-3t}$.

For $2t^4$ the associated root is 0
For t^2e^{-3t} the associated root is -3
For $\sin 3t$ the associated root(s) are $\pm 3\mathbf{i}$.
First version of test function is $\bar{Y}(t) =$

$$(At^4+Bt^3+Ct^2+Dt+E)+(Ft^2+Gt+H)e^{-3t}+(I\cos 3t+J\sin 3t).$$

Because 0 and -3 are roots from the homogeneous equation, each of those blocks must be multiplied by t to get $Y(t) =$

$$t(At^4+Bt^3+Ct^2+Dt+E)+t(Ft^2+Gt+H)e^{-3t}+(I\cos 3t+J\sin 3t).$$

Example 3.5/22 : $y'' + y = t(1 + \sin t) = t + t \sin t$.
Characteristic equation is $r^2 + 1 = 0$ with roots $r = \pm\mathbf{i}$.
$y_h = C_1 \cos t + C_2 \sin t$.

For t the associated root is 0.
For $t \sin t$ the associated root is $\pm\mathbf{i}$.
First version of test function is

$$\bar{Y}(t) \;=\; (At + B) + ((Ct + D)\cos t + (Et + F)\sin t).$$

Because $\pm\mathbf{i}$ are roots from the homogeneous equation, the corresponding block must be multiplied by t to get

$$Y(t) \;=\; (At + B) + t((Ct + D)\cos t + (Et + F)\sin t).$$

Example 3.5/25 :$y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t\sin 2t$.
Characteristic equation is $r^2 - 4r + 4 = (r-2)^2 = 0$ with roots $r = 2, 2$. $y_h = C_1 e^{2t} + C_2 te^{2t}$.

For $2t^2$ the associated root is 0.
For $4te^{2t}$ the associated root is 2.
For $t\sin 2t$ the associated root is $\pm 2\mathbf{i}$.
First version of test function is $\bar{Y}(t) =$

$$(At^2+Bt+C)+(Dt+E)e^{2t}+((Ft+G)\cos 2t+(Ht+I)\sin 2t).$$

Because 2 is a twice repeated root of the homogeneous equation, the corresponding block must multiplied by t^2. The test function $Y(t) =$

$$(At^2+Bt+C)+t^2(Dt+E)e^{2t}+((Ft+G)\cos 2t+(Ht+I)\sin 2t).$$

Example 4.2/11 : $y''' - y'' - y' + y = 0$. Characteristic equation is $r^3 - r^2 - r + 1 = 0$. Factor by grouping: $r^3 - r^2 - r + 1 = r^2(r-1) - (r-1) = (r^2-1)(r-1) = (r+1)(r-1)(r-1)$ with roots $-1, 1, 1$.

$$y_g = C_1 e^{-t} + C_2 e^t + C_3 t e^t.$$

Example 4.2/13 : $2y''' - 4y'' - 2y' + 4y = 0$. Divide the characteristic equation by 2 and factor by grouping to get $r^3 - 2r^2 - r + 2 = (r^2-1)(r-2)$ with roots $1, -1, 2$.

Example 4.2/15 : $y^{(6)} + y = 0$. The characteristic equation $r^6 = -1$ requires DeMoivre's Theorem. The modulus is $1^{1/6} = 1$. Each step is 60°, beginning with half-step 30°. The six roots are

$$(\cos 30) \pm \mathbf{i}(\sin 30) = (\sqrt{3}/2) \pm \mathbf{i}(1/2), (\cos 90) \pm \mathbf{i}(\sin 90) = \pm \mathbf{i},$$
$$(\cos 150) \pm \mathbf{i}(\sin 150) = (-\cos 30) \pm \mathbf{i}(\sin 30) = (-\sqrt{3}/2) \pm \mathbf{i}(1/2).$$

$$\begin{aligned} y_g = {} & C_1 e^{t\sqrt{3}/2}\cos(t/2) + C_2 e^{t\sqrt{3}/2}\sin(t/2) + C_3\cos(t) + C_4\sin(t) \\ & + C_5 e^{-t\sqrt{3}/2}\cos(t/2) + C_6 e^{-t\sqrt{3}/2}\sin(t/2). \end{aligned}$$

Example 4.2/21 : $y^{(8)} + 8y^{(4)} + 16y = 0$. The characteristic equation is $r^8 + 8r^4 + 16 = (r^4 + 4)^2 = 0$. $r^4 = -4$ requires DeMoivre's Theorem. The modulus is $4^{1/4} = \sqrt{2}$. Each step is 90°, beginning with half-step 45°. The four roots are

$$\sqrt{2}((\cos 45) \pm \mathbf{i}(\sin 45)) = 1 \pm \mathbf{i},$$
$$\sqrt{2}((\cos 135) \pm \mathbf{i}(\sin 135)) = \sqrt{2}(-(\cos 45) \pm \mathbf{i}(\sin 45)) = -1 \pm \mathbf{i}.$$

Each pair of complex roots is repeated.

$$y_g = C_1 e^t \cos t + C_2 e^t \sin t + C_3 te^t \cos t + C_4 te^t \sin t + C_5 e^{-t} \cos t + C_6 e^{-t} \sin t + C_7 te^{-t} \cos t + C_8 te^{-t} \sin t$$

Example 4.3/13 : $y''' - 2y'' + y' = t^3 + 2e^t$. The characteristic equation for the homogeneous equation is $r^3 - 2r^2 + r = r(r-1)^2 = 0$ with roots $0, 1, 1$.

$$y_h = C_1 + C_2 e^t + C_3 t e^t.$$

For t^3 the associated root is 0.
For $2e^t$ the associated root is 1.
The first guess for the test function Y_p^1 is thus

$$Y_p^1 = (At^3 + Bt^2 + Ct + D) \ + \ (Ee^t).$$

Because 0 is a root for the homogeneous equation, we must multiply the first block by t.
Because 1 is a repeated root for the homogeneous equation, we must multiply the second block by t^2.

$$Y_p = t(At^3 + Bt^2 + Ct + D) \ + \ t^2(Ee^t).$$

Example 4.3/15 : $y^{(4)} - 2y'' + y = e^t + \sin t$. The characteristic equation for the homogeneous equation is $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = 0$ with roots $1, 1, -1, -1$.

$$y_h = C_1 e^t + C_2 te^t + C_3 e^{-t} + C_4 te^{-t}.$$

For e^t the associated root is 1.
For $\sin t$ the associated root is the complex pair $\pm\mathbf{i}$.
The first guess for the test function Y_p^1 is thus

$$Y_p^1 = \; (Ae^t) \; + \; (B\cos(t) + C\sin(t)).$$

Because 1 is a repeated root for the homogeneous equation, we must multiply the first block by t^2.

$$Y_p = \; t^2(Ae^t) \; + \; (B\cos(t) + C\sin(t)).$$

Example 4.3/16 : $y^{(4)} + 4y'' = \sin 2t + te^t + 4$. The characteristic equation for the homogeneous equation is $r^4 + 4r^2 = r^2(r^2+4)$ with roots $0, 0, \pm 2\mathbf{i}$.

$$y_h = C_1 + C_2 t + C_3 \cos(2t) + C_4 \sin(2t).$$

For $\sin 2t$ the associated root is the complex pair $\pm 2\mathbf{i}$.
For te^t the associated root is 1.
For 4 the associated root is 0.
The first guess for the test function Y_p^1 is thus

$$Y_p^1 = (A\cos(2t) + B\sin(2t)) \; + \; (Ct + D)e^t \; + \; (F).$$

Because $\pm i$ are roots for the homogeneous equation, we must multiply the first block by t.
Because 0 is a repeated root for the homogeneous equation, we must multiply the third block by t^2.

$$Y_p = t(A\cos(2t) + B\sin(2t)) \; + \; (Ct + D)e^t \; + \; t^2(F).$$

Example 4.3/18 : $y^{(4)} + 2y''' + 2y'' = 3e^t + 2te^{-t} + e^{-t}\sin t$.
The characteristic equation for the homogeneous equation is $r^4 + 2r^3 + 2r^2 = r^2(r^2 + 2r + 2) = 0$. For the quadratic we use the Quadratic Formula, to get roots $0, 0, -1 \pm \mathbf{i}$.

$$y_h = C_1 + C_2 t + C_3 e^{-t}\cos(t) + C_4 e^{-t}\sin(t).$$

For $3e^t$ the associated root is 1.
For $2te^{-t}$ the associated root is -1.
For $e^{-t}\sin t$ the associated root is $-1 \pm \mathbf{i}$.
The first guess for the test function Y_p^1 is thus

$$Y_p^1 = (Ae^t) \; + \; (Bt + C)e^{-t} \; + \; (De^{-t}\cos(t) + Ee^{-t}\sin(2t)).$$

Because $-1 \pm \mathbf{i}$ are roots for the homogeneous equation, we must multiply the third block by t.

$$Y_p = (Ae^t) \; + \; (Bt + C)e^{-t} \; + \; t(De^{-t}\cos(t) + Ee^{-t}\sin(2t)).$$

Example 3.6/3 : $y'' + 2y' + y = 3e^{-t}$. The roots of the characteristic equation for the homogeneous equation $r^2 + 2r + 1 = (r+1)^2 = 0$ are $-1, -1$ and so $y_h = C_1 e^{-t} + C_2 t e^{-t}$.

Undetermined Coefficients:
The associated root for $3e^{-t}$ is -1. Our first guess for test function $Y_p^1 = Ae^{-t}$.
Because -1 is a root repeated twice, we must multiply by t^2.
So $Y_p = At^2 e^{-t}$.

$$
\begin{aligned}
1 \times Y_p &= At^2 e^{-t}, \\
2 \times Y_p' &= -2At^2 e^{-t} + 4Ate^{-t}, \\
1 \times Y_p'' &= At^2 e^{-t} - 4Ate^{-t} + 2Ae^{-t}.
\end{aligned}
$$

So $3e^{-t} = 2Ae^{-t}$. So $A = 3/2$ and
$y_g = C_1 e^{-t} + C_2 t e^{-t} + (3/2) t^2 e^{-t}$.

Variation of Parameters:
We look for $y_p = u_1 e^{-t} + u_2 t e^{-t}$. We have the linear equations:

$$\begin{array}{rcl} u_1'(e^{-t}) + u_2'(te^{-t}) & = & 0 \\ u_1'(-e^{-t}) + u_2'(e^{-t} - te^{-t}) & = & 3e^{-t}. \end{array}$$

Wronskian is e^{-2t} so
$u_1' = \begin{vmatrix} 0 & te^{-t} \\ 3e^{-t} & (e^{-t} - te^{-t}) \end{vmatrix} / e^{-2t}$,

$u_2' = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & (3e^{-t}) \end{vmatrix} / e^{-2t}$.
$u_1' = -3t, u_2' = 3$. $u_1 = -(3/2)t^2, u_2 = 3t$.

$$Y_p = -(3/2)t^2 e^{-t} + (3t)te^{-t} = (3/2)t^2 e^{-t}.$$

Example 3.6/5 : $y'' + y = \tan(t)$. The roots of the characteristic equation for the homogeneous equation $r^2 + 1 = 0$ are $\pm\mathbf{i}$ and so $y_h = C_1 \cos(t) + C_2 \sin(t)$.

We look for $y_p = u_1 \cos(t) + u_2 \sin(t)$. We have the linear equations:

$$\begin{array}{rcrcl} u_1'(\cos(t)) & + & u_2'(\sin(t)) & = & 0 \\ u_1'(-\sin(t)) & + & u_2'(\cos(t)) & = & \tan(t). \end{array}$$

Wronskian is 1.
$u_1' = -\sin(t)\tan(t) = -\sin^2(t)/\cos(t) =$
$(\cos^2(t) - 1)/\cos(t) = \cos(t) - \sec(t)$,
$u_2' = \cos(t)\tan(t) = \sin(t)$.
$u_1 = \sin(t) - \ln|\sec(t) + \tan(t)|$, $u_2 = -\cos(t)$,

$$y_p = -\sin(t)\ln|\sec(t) + \tan(t)|.$$

Example 3.6/10 : $y'' - 2y' + y = e^t/(1+t^2)$. The roots of the characteristic equation for the homogeneous equation $r^2 - 2r + 1 = 0$ are $1, 1$ and so $y_h = C_1 e^t + C_2 te^t$.
$y_p = u_1 e^t + u_2 te^t$

$$\begin{aligned} u_1'(e^t) \ + \ u_2'(te^t) \qquad &= \quad 0 \\ u_1'(e^t) \ + \ u_2'(e^t + te^t) \quad &= \quad e^t/(1+t^2). \end{aligned}$$

Wronskian is e^{2t}.
$u_1' = -t/(1+t^2), u_2' = 1/(1+t^2)$.
$u_1 = -\frac{1}{2}\ln(1+t^2), u_2 = \arctan(t)$.

$$y_p = -\frac{1}{2}e^t \ln(1+t^2) \ + \ te^t \arctan(t).$$

Example 3.6/14 : $t^2y'' - t(t+2)y + (t+2)y = 2t^3$. Divide by t^2 to get
$y'' - t^{-1}(t+2)y + t^{-2}(t+2)y = 2t$.
Given $y_1 = t$, $Y_2 = te^t$. (Check that these are solutions).

$y_p = u_1 t + u_2 te^t$.

$$\begin{array}{rcl} u_1'(t) \; + \; u_2'(te^t) & = & 0 \\ u_1'(1) \; + \; u_2'(e^t + te^t) & = & 2t. \end{array}$$

Wronskian is t^2e^t.
$u_1' = -2t^2e^t/t^2e^t = -2, u_2' = 2t^2/t^2e^t = 2e^-t$.
$u_1 = -2t, u_2 = -2e^{-t}$.

$$y_p = -2t^2 - 2t.$$

Example 3.7/6 : Units are centimeters, grams and seconds.
So $g = 980 cm/s^2$. $m = 100 gr$, $\Delta L = 5cm$,
From equilibrium, so $y_0 = 0$.
Downward velocity of $10cm/s$, so $y_0' = -10$.
$w = mg = 98000$ (dynes).$w = k\Delta L$, so
$k = 98000/5 = 19600$.
No damping, so $c = 0$. No external force.

$$100y'' + 19600y = 0, \qquad \text{or} \quad y'' + 196y = 0,$$

Characteristic equation: $r^2 + 196 = 0$ with roots $r = \pm 14\mathbf{i}$. So

$$\begin{aligned} y &= C_1 \cos 14t + C_2 \sin 14t \\ y' &= 14C_2 \cos 14t - 14C_1 \sin 14t. \end{aligned}$$

$0 = y(0) = C_1, \quad -10 = y'(0) = 14C_2.$
So the solution is $y = -(5/7) \sin 14t$.

Example 3.7/7 : Units are feet, pounds and seconds.
So $g = 32ft/s^2$. $w = 3lb$, $\Delta L = 3/12 = 1/4ft$,
Lifted up, so $y_0 = 1/12ft$.
Downward velocity of $2ft/s$, so $y_0' = -2$. $3w = mg = 32m$, so $m = 3/32$ (slugs).$w = k\Delta L$, so $k = 3 \div (1/4) = 12$.
No damping, so $c = 0$. No external force.

$$3y'' + 12y = 0, \qquad \text{or} \quad y'' + 4y = 0,$$

Characteristic equation: $r^2 + 4 = 0$ with roots $r = \pm 2\mathbf{i}$. So

$$\begin{aligned} y &= C_1 \cos 2t + C_2 \sin 2t \\ y' &= 2C_2 \cos 2t - 2C_1 \sin 2t. \end{aligned}$$

$-2 = y(0) = C_1, \quad -2 = y'(0) = 2C_2.$
So the solution is $y = -2\cos 2t + 1 \sin 2t$.
$A = \sqrt{5}, \phi = 150^\circ = 5\pi/6 rad$
$y = \sqrt{5}\cos(2t - 5\pi/6)$. [This part I won't ask.]

Example 3.7/9 : Units are centimeters, grams and seconds with $g = 980 cm/s^2$.
$m = 20gr$, $\Delta L = 5cm$, $c = 400$
Down $2cm$, so $y_0 = -2$. Released, so so $y_0' = 0$.
$w = mg = 19600$ (dynes).$w = k\Delta L$, so $k = 19600/5 = 3920$.
No external force.

$$20y'' + 400y' + 3920y = 0, \qquad \text{or} \quad y'' + 20y' + 196y = 0,$$

Characteristic equation: $r^2 + 20r + 196 = 0$ with roots

$$r \;=\; (-\frac{\sqrt{20}}{2}) \pm \frac{\sqrt{384}}{2}\mathbf{i} \;=\; -10 \pm 4\sqrt{6}\mathbf{i}.$$

So

$$\begin{aligned} y &= C_1 e^{-10t}\cos(4\sqrt{6}t) \quad + \quad C_2 e^{-10t}\sin(4\sqrt{6}t) \\ y' &= (-10C_1 + 4\sqrt{6}C_2)e^{-10t}\cos(4\sqrt{6}t) + \\ &\qquad\qquad (-10C_2 - 4\sqrt{6}C_1)e^{-10t}\sin(4\sqrt{6}t). \end{aligned}$$

$-2 = y(0) = C_1, \quad 0 = y'(0) = -10C_1 + 4\sqrt{6}C_2.$
So the solution is

$$y = -2e^{-10t}\cos(4\sqrt{6}t) - \frac{5}{\sqrt{6}}e^{-10t}\sin(4\sqrt{6}t).$$

Example 3.8/10 : Units are feet, pounds and seconds.
$w = 8lb$. So $m = w/g = 8/32 = 1/4$. $\Delta L = 1/2$. So
$k = w/\Delta L = 16$. $c = 0$.
$y(0) = -1/4, y'(0) = 0$.
$(1/4)y'' + 16y = 8\sin(8t)$, or $\quad y'' + 64y = 32\sin(8t)$.
The roots of the characteristic equation for the homogeneous equation $r^2 + 64 = 0$ are $\pm 8\mathbf{i}$. So the forcing occurs at resonance.
$Y_p = At\cos(8t) + Bt\sin(8t)$
$Y_p' = -8At\sin(8t) + 8Bt\cos(8t) + A\cos(8t) + B\sin(8t)$.
$Y_p'' = -64At\cos(8t) - 64At\sin(8t) - 16A\sin(8t) + 16B\cos(8t)$.
$32\sin(8t) = Y_p'' + 64Y_p = -16A\sin(8t) + 16B\cos(8t)$. So
$A = -2, B = 0$.
$y = C_1\cos(8t) + C_2\sin(8t) - 2t\cos(8t)$,
$y' = -8C_1\sin(8t) + 8C_2\cos(8t) - 2\cos(8t) + 16t\sin(8t)$.
At $t = 0$: $-1/4 = C_1, 0 = 8C_2 - 2$. So

$$y = -(1/4)\cos(8t) + (1/4)\sin(8t) - 2t\cos(8t).$$

Example 5.2/9 : $(1+x^2)y'' - 4xy' + 6y = 0$

$$1y'' = \Sigma\ 1n(n-1)a_n x^{n-2}[k = n-2] = \Sigma(k+2)(k+1)a_{k+2}x^k.$$
$$+x^2y'' = \Sigma\ n(n-1)a_n x^n\ [k=n] = \Sigma\ k(k-1)a_k x^k.$$
$$-4xy' \quad = \quad \Sigma 4na_n x^n \quad [k=n] = \quad \Sigma\ -4ka_k x^k.$$
$$6y \quad = \quad \Sigma\ 6a_n x^n \qquad [k=n] = \quad \Sigma\ 6a_k x^k.$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[-(k(k-1)+4k-6)a_k] = \frac{-k^2+5k-6}{(k+2)(k+1)}a_k$$

$$= -\frac{(k-2)(k-3)}{(k+2)(k+1)}a_k.$$

$k=0 : a_2 = -3a_0, \qquad k=1 : a_3 = -\frac{1}{3}a_1.$
$k=2 : a_4 = 0, \qquad k=3 : a_5 = 0.$ So $a_k = 0$ for $k \geq 2$.

Example 5.2/12,18:
$(1-x)y'' + xy' + -y = 0, y(0) = -3, y'(0) = 2.$

$$1y'' = \Sigma\ 1n(n-1)a_n x^{n-2}[k=n-2] = \Sigma(k+2)(k+1)a_{k+2}x^k$$
$$-xy'' = \Sigma\ -n(n-1)a_n x^{n-1}\ [k=n-1] = \Sigma\ -(k+1)(k)a_{k+1}x^k.$$
$$+xy' = \Sigma\ na_n x^n\ [k=n] = \Sigma\ ka_k x^k.$$
$$-y = \Sigma\ -a_n x^n\ [k=n] = \Sigma\ -a_k x^k.$$

$$a_{k+2} = \frac{1}{(k+2)(k+1)}[(k+1)ka_{k+1} - (k-1)a_k], \quad a_0 = -3, a_1 = 2.$$

$k = 0 : a_2 = a_0/2 = -3/2,$
$k = 1 : a_3 = a_2/3 = -1/2.$
$k = 2 : a_4 = [6a_3 - a_2]/12 = -3/4,$
$k = 3 : a_5 = [12a_4 - 2a_3]/20 = -2/5.$