Math 391 Test 2A
April Fool's Day 2015

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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.

2. Don't panic! I repeat, do NOT panic!

3. Complete all problems. In this exam, each non-bonus problem is worth 20 points. The weight of the bonus problems are indicated.

4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)

5. Write neatly so that I am able to follow your sequence of steps and box your answers.

6. Read through the exam and complete the problems that are easy (for you) first!

7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.

8. In fact, **cell phones should be out of sight!**

9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.

10. Do NOT commit any of the blasphemies or mistakes I mentioned in the syllabus. I will actually mete out punishment in the way I said I would. I wasn't kidding.

11. Other than that, have fun and good luck!

Remember: This is the test this class deserves, and also the one it needs right now.

Now, if my calculations are correct, when this baby hits 88 miles an hour, you're going to see some serious s*@!#!

1. Solve the following ODEs:

(a) $6y'' - y' - 2y = 0$

$6r^2 - r - 2 = 0$

$\Rightarrow (3r-2)(2r+1) = 0$

$\Rightarrow r = \frac{2}{3},\ r = -1/2$

$\Rightarrow \boxed{y = C_1 e^{2t/3} + C_2 e^{-t/2}}$

(b) $4y'' - 12y' + 9y = 0$

$4r^2 - 12r + 9 = 0$

$\Rightarrow (2r-3)^2 = 0$

$\Rightarrow r_{1,2} = 3/2$

$\Rightarrow \boxed{y = C_1 e^{3t/2} + C_2 t e^{3t/2}}$

(c) $y'' + 4y' + 7y = 0$

$r^2 + 4r + 7 = 0$

$$\Rightarrow r = \frac{-4 \pm \sqrt{16 - 4(7)}}{2}$$

$$= \frac{-4 \pm 2\sqrt{-3}}{2}$$

$$= -2 \pm \sqrt{3}\, i$$

$\Rightarrow \boxed{y = C_1 e^{-2t} \cos\sqrt{3}t + C_2 e^{-2t} \sin\sqrt{3}t}$

2. Given the differential equation $2t^2y'' + ty' - y = 0, t > 0$; notice that $y_1 = t^{-1/2}$ is a solution. Use reduction of order to find a second, linearly independent solution, y_2. *Prove that your second solution is linearly independent to the first.*

Assume $y_2 = vy_1$

$= vt^{-1/2}$

$\Rightarrow y_2' = v't^{-1/2} - \frac{1}{2}vt^{-3/2}$

$\Rightarrow y_2'' = v''t^{-1/2} - \frac{1}{2}v't^{-3/2} - \frac{1}{2}v't^{-3/2} + \frac{3}{4}vt^{-5/2}$

$= v''t^{-1/2} - v't^{-3/2} + \frac{3}{4}vt^{-5/2}$

Plug these into the ODE.

$\Rightarrow 2t^2(v''t^{-1/2} - v't^{-3/2} + \frac{3}{4}vt^{-5/2}) + t(v't^{-1/2} - \frac{1}{2}vt^{-3/2}) - vt^{-1/2} = 0$

$\Rightarrow 2v''t^{3/2} - 2v't^{1/2} + \cancel{\frac{3}{2}vt^{-1/2}} + v't^{1/2} - \cancel{\frac{1}{2}vt^{-1/2}} - \cancel{vt^{-1/2}} = 0$

$\Rightarrow 2v''t^{3/2} - v't^{1/2} = 0$

$\Rightarrow 2v''t - v' = 0$ (since $t \neq 0$)

Let $u = v'$

$\Rightarrow u' = v''$

$\Rightarrow 2u't - u = 0$

$\Rightarrow \frac{du}{u} = \frac{dt}{2t}$

$\Rightarrow \ln|u| = \frac{1}{2}\ln|t| + C$

$\Rightarrow u = Ct^{1/2}$ (since $t > 0$)

$\Rightarrow v' = Ct^{1/2}$

$\Rightarrow v = \frac{2}{3}Ct^{3/2} + D$

$\Rightarrow y_2 = \frac{2}{3}Ct + Dt^{-1/2}$

Take $\boxed{y_2 = t}$

To prove they are linearly independent

$$W = \begin{vmatrix} t^{-1/2} & t \\ -\frac{1}{2}t^{-3/2} & 1 \end{vmatrix}$$

$= t^{-1/2} + \frac{1}{2}t^{-1/2}$

$= \frac{3}{2}t^{-1/2}$

$\neq 0$

since $t > 0$

$\Rightarrow \boxed{y_1, y_2 \text{ are linearly indep.}}$

$m = \frac{4}{32} = \frac{1}{8}$, $\gamma = \frac{4}{2} = 2$, $k = \frac{4}{\frac{1}{4}} = 16$

3. A mass weighing 4 lbs stretches a spring 3 in. The damping is such that the resistance force is 4 lbs when the mass is traveling at 2 ft/sec. Starting from the equilibrium position, the mass is set in motion with a downward velocity of 8 ft/sec.

(a) Set up a differential equation, with initial conditions, for the function $u(t)$ that models the motion of the mass.

$$\boxed{\tfrac{1}{8}u'' + 2u' + 16u = 0, \quad u(0) = 0, \quad u'(0) = 8}$$

(b) Solve the differential equation in part (a).

$$\tfrac{1}{8}r^2 + 2r + 16 = 0$$

$$\Rightarrow r = \frac{-2 \pm \sqrt{4 - 4(\frac{1}{8})(16)^2}}{\frac{1}{4}}$$

$$\Rightarrow r = 4\left(-2 \pm 2\sqrt{-1}\right)$$

$$\Rightarrow r = -8 \pm 8i$$

$$\Rightarrow u = C_1 e^{-8t}\cos 8t + C_2 e^{-8t}\sin 8t$$

$$\Rightarrow u' = -8C_1 e^{-8t}\cos 8t - 8C_1 e^{-8t}\sin 8t - 8C_2 e^{-8t}\sin 8t + 8C_2 e^{-8t}\cos 8t$$

$u(0) = 0 = C_1$

$u'(0) = 8 = 8C_2 \Rightarrow C_2 = 1$

$$\Rightarrow \boxed{u(t) = e^{-8t}\sin 8t}$$

(c) At what time does the mass first return to the equilibrium position? Justify. In what direction is the mass moving in at this time?

We need $u(t) = e^{-8t}\sin 8t = 0$

$$\Rightarrow \sin 8t = 0$$

$$\Rightarrow t = \frac{n\pi}{8} = 0, \frac{\pi}{8}, \frac{\pi}{4}, \ldots$$

1st time to return.

$\boxed{t = \frac{\pi}{8} \text{ seconds}}$ • $\boxed{\text{The mass is moving upwards!}}$

4. Solve the ODE: $y'' + y = \sec t$, $0 \le t < \frac{\pi}{2}$, subject to $y(0) = 3, y'(0) = 7$.

↳ Variation of parameters!

$r^2 + 1 = 0$

$\Rightarrow r = \pm i$

$\Rightarrow y_1 = \cos t, \quad y_2 = \sin t \quad (\text{since } y_h = c_1 \cos t + c_2 \sin t)$

$$\Rightarrow W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$\Rightarrow y_p = -y_1 \int \frac{y_2 \cdot g}{W} + y_2 \int \frac{y_1 \cdot g}{W}$$

$$= -\cos t \int \frac{\sin t \cdot \sec t}{1} dt + \sin t \int \frac{\cos t \sec t}{1} dt$$

$$= -\cos t \int \tan t \, dt + \sin t \int 1 \, dt$$

$$= \cos t \ln|\cos t| + t \sin t$$

$\Rightarrow y = c_1 \cos t + c_2 \sin t + \cos t \ln|\cos t| + t \sin t$

$\Rightarrow y' = -c_1 \sin t + c_2 \cos t - \sin t \ln|\cos t| + \cos t \cdot -\tan t + \sin t + t \cos t$

$y(0) = 3 = c_1$

$y'(0) = 7 = c_2$

$$\Rightarrow \boxed{y = 3\cos t + 7 \sin t + \cos t \ln|\cos t| + t \sin t}$$

5. Solve the ODE: $y'' - 6y' + 9y = t + e^{3t}$

$$r^2 - 6r + 9 = 0$$

$$\Rightarrow (r-3)^2 = 0$$

$$\Rightarrow r_{1,2} = 3$$

$$\Rightarrow y_h = c_1 e^{3t} + c_2 t e^{3t}$$

$$y_p = At + B + Ct^2 e^{3t}$$

$$\Rightarrow y_p' = A + 2Cte^{3t} + 3Ct^2e^{3t} = A + (2Ct + 3Ct^2)e^{3t}$$

$$\Rightarrow y_p'' = (2C + 6Ct)e^{3t} + 3(2Ct + 3Ct^2)e^{3t}$$

plug this into the ODE

$$\Rightarrow (2C+6Ct)e^{3t} + 3(2Ct+3Ct^2)e^{3t} - 6A - 6(2Ct+3Ct^2)e^{3t} + 9At + 9B + 9Ct^2e^{3t}$$
$$= t + e^{3t}$$

$$\rightarrow e^{3t}(2C) + te^{3t}(6C+6C-12C) + t^2e^{3t}(9C-18C+9C) + t(9A) + 9B - 6A = t + e^{3t}$$

$$\Rightarrow 2c = 1 \Rightarrow c = 1/2$$

$$9A = 1 \Rightarrow A = 1/9$$

$$9B - 6A = 0$$

$$\Rightarrow 9B - \frac{2}{3} = 0$$

$$\Rightarrow B = \frac{2}{27}$$

$$\Rightarrow \boxed{y = c_1e^{3t} + c_2te^{3t} + \frac{1}{9}t + \frac{2}{27} + \frac{1}{2}t^2e^{3t}}$$

Bonus Problems:

1. (2 points) Find the general solution of $y''' + 3y'' + 3y' + y = 0$

$r^3 + 3r^2 + 3r + 1 = 0$

$\Rightarrow (r+1)^3 = 0$

$\Rightarrow r_{1,2,3} = -1$

$$\Rightarrow \boxed{y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}}$$

2. (8 points) Find the general solution of $y^{(6)} + y = 0$

$r^6 + 1 = 0$

$\Rightarrow r^6 = -1$

$= e^{i(\pi + 2n\pi)}$

$\Rightarrow r = e^{i(\pi+2n\pi)/6}, \quad n = 0, 1, 2, \cdots, 5$

$\Rightarrow r_1 = e^{i\pi/6} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$r_2 = e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$r_3 = e^{i5\pi/6} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$

$\Rightarrow r_4 = \frac{\sqrt{3}}{2} - i\frac{1}{2}, \quad r_5 = -i, \quad r_6 = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$ (since complex roots come in conjugate pairs).

$$\Rightarrow \boxed{y = c_1\cos t + c_2\sin t + c_3 e^{\sqrt{3}t/2}\cos\frac{t}{2} + c_4 e^{\sqrt{3}t/2}\sin\frac{t}{2} + c_5 e^{-\sqrt{3}t/2}\cos\frac{t}{2} + c_6 e^{-\sqrt{3}t/2}\sin\frac{t}{2}}$$

3. (10 points) Find the general solution of $y^{(4)} - 3y''' + y'' + 5y' = 0$

$r^4 - 3r^3 + r^2 + 5r = 0$

$\Rightarrow r(r^3 - 3r^2 + r + 5) = 0$

$r_1 = 0$ or $r^3 - 3r^2 + r + 5 = 0$

$r_2 = -1$ works.

$$\begin{array}{r} r^2 - 4r + 5 \\ r+1 \overline{) r^3 - 3r^2 + r + 5} \\ -(r^3 + r^2) \\ -4r^2 \\ -(-4r^2 - 4r) \\ 5r \\ -(5r+5) \\ 0 \end{array}$$

$\Rightarrow r = \dfrac{4 \pm \sqrt{16 - 20}}{2}$

$r_{3,4} = 2 \pm i$

$$\Rightarrow \boxed{y = C_1 + C_2 e^{-t} + C_3 e^{2t} \cos t + C_4 e^{2t} \sin t}$$

4. (5 points) Write down the form for the general solution to $y^{(6)} + 2y''' + y = t^2 + e^{-t} + 5e^{t/2} \sin \frac{\sqrt{3}t}{2}$. For your y_p, you need not find the arbitrary constants, but it must be minimal and have the fewest terms possible.

$r^6 + 2r^3 + 1 = 0$

$(r^3 + 1)^2 = 0$

$\Rightarrow r^3 + 1 = 0$

$\Rightarrow (r+1)(r^2 - r + 1) = 0$

$r = -1, \; r = \dfrac{1 \pm \sqrt{1-4}}{2}$

$= \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\Rightarrow y_h = c_1 e^{-t} + c_2 e^{t/2} \cos \frac{\sqrt{3}}{2}t + c_3 e^{t/2} \sin \frac{\sqrt{3}}{2}t + c_4 t e^{-t} + c_5 t e^{t/2} \cos \frac{\sqrt{3}}{2}t + c_6 t e^{t/2} \sin \frac{\sqrt{3}}{2}t$

$$\Rightarrow \boxed{y_p = At^2 + Bt + C + Dt^2 e^{-t} + Et^2 e^{t/2} \cos \tfrac{\sqrt{3}}{2}t + Ft^2 e^{t/2} \sin \tfrac{\sqrt{3}}{2}t}$$