Math 391 Test 1A
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Note that both sides of each page may have printed material.

Instructions:

1. Read the instructions.
2. Panic!!! Kidding, don't panic! I repeat, do NOT panic!
3. Complete all problems. In this exam, each non-bonus problem is worth 20 points. The weight of the bonus problems are indicated.
4. Show ALL your work to receive full credit. You will get 0 credit for simply writing down the answers (unless it's a case of fill in the blank or state a definition, etc.)
5. Write neatly so that I am able to follow your sequence of steps and box your answers.
6. Read through the exam and complete the problems that are easy (for you) first!
7. No scrap paper, calculators, notes or other outside aids allowed—including divine intervention, telepathy, knowledge osmosis, the smart kid that may be sitting beside you or that friend you might be thinking of texting.
8. In fact, **cell phones should be out of sight!**
9. Use the correct notation and write what you mean! x^2 and $x2$ are not the same thing, for example, and I will grade accordingly.
10. Do NOT commit any of the blasphemies or mistakes I mentioned in the syllabus. I will actually mete out punishment in the way I said I would. I wasn't kidding.
11. Other than that, have fun and good luck!

Remember: math is fun, math is beautiful, this test is *not* hard, there is no spoon.

1.

(a) Write down the standard form for a first order linear ODE: $y' + p(t)y = g(t)$

(b) Write down a general form of an exact equation: $M(x,y)\,dx + N(x,y)\,dy = 0$

(c) What condition on the above equation would make it exact? $M_y = N_x$

(d) Let $Q(t)$ be the amount of a pollutant in a tank at time t, let Q_0 be the initial amount of pollutant in the tank, c_{in} be the concentration of pollutant flowing into the tank at a rate r_{in}, and let V_0 be the initial volume in the tank. Let r_{out} be the rate at which the mixture in the tank flows out. Write down the differential equation with initial condition to describe $Q(t)$.

ODE: $Q' = c_{in} r_{in} - \frac{Q}{V_0 + (r_{in} - r_{out})t} \cdot r_{out}$, initial condition: $Q(0) = Q_0$

(e) Write down a general form for a separable first order ODE: $f(y)\,dy = g(t)\,dt$

(f) What condition would make $\frac{dy}{dx} = f(x,y)$ homogeneous? $f(kx,ky) = f(x,y)$ or $f(x,y) = g\left(\frac{y}{x}\right)$ for some function g.

(g) Find the general solution of the following:

(i) $\frac{dy}{dx} = \frac{1-2xy-x^2y}{x^2}$, $x > 0$

(This is linear!)

$\Rightarrow y' + \left(1 + \frac{2}{x}\right)y = \frac{1}{x^2}$

$\mu = e^{\int 1 + \frac{2}{x}dx} = e^{x + \ln|x|^2} = x^2 e^x$

$\Rightarrow \int [x^2 e^x y]' = \int e^x$

$\Rightarrow x^2 e^x y = e^x + C$

$\Rightarrow \boxed{y = \frac{1}{x^2} + \frac{C}{x^2 e^x}}$

(ii) $x\,dx + ye^{-x}dy = 0$

(This is separable!)

$ye^{-x}dy = -x\,dx$

$\Rightarrow \int y\,dy = -\int xe^x\,dx$

$\Rightarrow \frac{y^2}{2} = -xe^x + e^x + C$

$\Rightarrow \boxed{y^2 = -2xe^x + 2e^x + C}$

2. Find the general solution of the following:

(a) $(4x^2 + 5xy + y^2)dx - x^2 dy = 0$

(This is homogeous!)

$$\Rightarrow \frac{dy}{dx} = \frac{4x^2 + 5xy + y^2}{x^2}, \text{ set } v = \frac{y}{x}, \text{ we get}$$

$$\Rightarrow v + x\frac{dv}{dx} = 4 + 5v + v^2$$

$$\Rightarrow x\frac{dv}{dx} = 4 + 4v + v^2$$

$$\Rightarrow \int \frac{1}{4+4v+v^2}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{1}{(v+2)^2}\,dv = \ln|x| + C$$

$$\Rightarrow -\frac{1}{v+2} = \ln|x| + C$$

$$\Rightarrow v = -2 - \frac{1}{\ln|x| + C}$$

$$\Rightarrow \boxed{y = -2x - \frac{x}{\ln|x| + C}}$$

(b) $\underbrace{(e^x \cos(xy) - ye^x \sin(xy) + 2x + y)}_{M}dx - \underbrace{(xe^x \sin(xy) + 2y - x)}_{N}dy = 0$

$$\left.\begin{aligned} M_y &= -xe^x \sin xy - e^x \sin xy - xye^{xy}\cos xy + 1 \\ N_x &= -(e^x \sin xy + xe^x \sin xy + xye^{xy}\cos xy - 1) \end{aligned}\right\} \text{Exact!}$$

$$\Rightarrow \Psi = \int M\,dx = \int \underbrace{e^x\cos xy - ye^x \sin xy}_{\frac{d}{dx} e^x \cos xy} + 2x + y\,dx$$

a very similar integral was done in class and I mentioned this "trick".

$$= e^x \cos xy + x^2 + xy + f(y).$$

also $\Psi = \int N\,dy = \int -xe^x \sin xy - 2y + x\,dy$

$$= e^x \cos xy - y^2 + xy + g(x).$$

$$\Rightarrow \boxed{e^x \cos xy + x^2 - y^2 + xy = C}$$

↳ General soln.

3. Marty was voted the class of 1985's "Most Likely to Abuse Time Travel"—a prediction that was realized shortly after he married his high school sweetheart. He goes back to the future and, unfortunately, discovers that he will die at age 85. Knowing that he will retire at age 65 (being the abuser of time that he is), he decides to go back to the present and save up enough money to make it to age 85 comfortably. When he retires at 65, he plans to put his savings into an account earning 5% interest, compounded continuously. He also plans to withdraw $80,000 per year to live on and enjoy his remaining days having adventures with Doc. Brown.

(a) Suppose $P(t)$ represents the balance of his account at time t (in years). Write down an equation for $\frac{dP}{dt}$.

$$\frac{dP}{dt} = 0.05P - 80000$$

(b) Solve the equation in part (a), assuming an initial investment of P_0 dollars.

$$\Rightarrow \frac{dP}{0.05P - 80000} = dt$$

$$\Rightarrow \frac{1}{0.05} \ln|0.05P - 80000| = t + C$$

$$\Rightarrow \ln|0.05P - 80000| = 0.05t + C$$

$$\Rightarrow P = \frac{Ce^{0.05t} + 80000}{0.05}$$

$$\Rightarrow P = Ce^{0.05t} + 1600000$$

$$P(0) = P_0 \Rightarrow P_0 = C + 1600000$$

$$\Rightarrow C = P_0 - 1600000$$

$$\Rightarrow \boxed{P = (P_0 - 1600000)e^{0.05t} + 1600000}$$

(c) How much should Marty invest at age 65, to have just enough money until he reaches age 85?

We want $P(20) = 0$.

$$\Rightarrow 0 = (P_0 - 1600000)e^{0.05(20)} + 1600000$$

$$\Rightarrow \boxed{P_0 = \frac{-1600000}{e} + 1600000}$$

FYI: This is about $1011392.89. Work hard, Marty!

4. A 200 gallon tank initially contains 100 gallons of fresh water. Water containing 1/5 lb of salt per gallon is pumped into the tank at a rate of 10 gallons per minute, and the mixture is pumped out of the tank at 5 gallons per minute. Suppose $Q(t)$ is the amount of salt in the tank at time t minutes.

(a) Set up a differential equation with initial condition to describe $Q(t)$ and solve it.

(The answer to 1(d) gives us the formula!)

$$Q' = \frac{1}{5}\cdot 10 - \frac{Q}{100+(10-5)t}\cdot 5 \quad , \quad Q(0)=0$$

$$\Rightarrow Q' = 2 - \frac{Q}{20+t}$$

$$\Rightarrow Q' + \frac{Q}{20+t} = 2 \quad \text{(This is linear!)}$$

$$\mu = e^{\int \frac{1}{20+t}dt} = e^{\ln|20+t|} = (20+t) \quad , \text{ since } t \geq 0.$$

$$\Rightarrow \int \left[(20+t)Q\right]' = \int 2(20+t)$$

$$\Rightarrow (20+t)Q = (20+t)^2 + C$$

$$\Rightarrow Q = 20+t+\frac{C}{20+t}$$

Since $Q(0)=0 \Rightarrow 0 = 20 + \frac{C}{20} \Rightarrow C = -400$

$$\Rightarrow \boxed{Q = 20+t-\frac{400}{20+t}}$$

(b) What will be the concentration of salt in the tank at the point of overflow?

Set V=volume $= 100+5t = 200 \Rightarrow t = 20$ mins $\rightarrow$ time to overflow.

$$\Rightarrow \text{Concentration} = \frac{Q(20)}{200} \text{ lb/gal}$$

$$= \frac{20+20-10}{200}$$

$$= \frac{3\cancel{0}}{20\cancel{0}}$$

$$= \boxed{\frac{3}{20} \text{ lb/gal}}$$

5. Find the general solution of:

(a) $ty'' + y' = 1, \qquad t > 0$

Set $v = y'$

$\Rightarrow tv' + v = 1$

$\Rightarrow \int (tv)' = \int 1$

$\Rightarrow tv = t + C$

$\Rightarrow v = 1 + \frac{C}{t}$

$\Rightarrow y' = 1 + \frac{C}{t}$

$\Rightarrow \boxed{y = t + C\ln|t| + D}$ $\quad C, D$ arbitrary constants.

(b) $y'' + y' = e^{-t}$

Set $v = y'$

$\Rightarrow v' + v = e^{-t}$

$\mu = e^{\int 1 dt} = e^{t}$

$\Rightarrow \int (e^{t} v)' = \int 1$

$\Rightarrow e^{t} v = t + C$

$\Rightarrow v = te^{-t} + Ce^{-t}$

$\Rightarrow y' = te^{-t} + Ce^{-t}$

$\Rightarrow y = -te^{-t} - e^{-t} - Ce^{-t} + D$

$-e^{-t} - Ce^{-t} = \underbrace{(-1-C)}_{\text{constant}} e^{-t}$

$\Rightarrow \boxed{y = -te^{-t} + Ce^{-t} + D}$

Bonus Problems:

1. (5 points) Find the general solution of $y'' - y' - 6y = 0$

(Assuming $y = e^{rt}$, we get...)

$$r^2 - r - 6 = 0$$

$$\Rightarrow (r-3)(r+2) = 0$$

$$\Rightarrow r = 3,\ r = -2$$

$$\Rightarrow \boxed{y = c_1 e^{3t} + c_2 e^{-2t}}$$

2. (5 points) Find the general solution of $x^2y'' - 2xy' - 10y = 0$

(Assuming $y = x^r$, we get...)

$$r(r-1) - 2r - 10 = 0$$

$$\Rightarrow r^2 - 3r - 10 = 0$$

$$\Rightarrow (r-5)(r+2) = 0$$

$$\Rightarrow r = 5,\ r = -2$$

$$\Rightarrow \boxed{y = c_1 x^5 + c_2 x^{-2}}$$

3. (5 points) Find the general solution of $y + (2x - ye^y)y' = 0$

Not linear, not homogeneous. $M_y \neq N_x$, so not exact. Lets try to make it exact. (This exact—pardon the pun—problem was done in class, see the lecture video for exact equations).

$$\frac{M_y - N_x}{N} = \frac{1-(2)}{2x - ye^y} \neq Q(x)$$

$$\frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y} = P(y)$$

$$\Rightarrow \mu = e^{\int \frac{1}{y}} = y$$

$$\Rightarrow y^2 + (2xy - y^2e^y)y' = 0.$$

$$\Rightarrow M_y = 2y,\ N_x = 2y = M_y$$

Exact!

$$\Rightarrow \psi = \int y^2\,dx = xy^2 + f(y)$$

$$\Rightarrow_{\text{also}} \psi = \int 2xy - y^2e^y\,dy = xy^2 - y^2e^y + 2ye^y - 2e^y + g(x)$$

$$\Rightarrow \boxed{xy^2 - y^2e^y + 2ye^y - 2e^y = c}$$

Rough work

$\int y^2 e^y\,dy$

$+$	$2y$	e^y	y^2e^y
$-$	2	e^y	$-2ye^y$
$+$	0	e^y	$+2e^y$