Math 37600 PR (19364) Test 1 Solutions Fall 2024

1-(12 points) (a) For a bounded, nonconstant, rv X, which is larger $E(X)^2$ or $E(X^2)$? Justify your answer in detail.

(b) Let X_1 and X_2 be independent -nonconstant- rv's with $E(X_1) = E(X_2) = 0$ and for $i = 1, 2$

$$E(X_i^2) = A_i, \quad E(X_i^3) = B_i, \quad E(X_i^4) = C_i.$$

Let $Z = X_1 + X_2$. Show that $E(Z^2) = A_1 + A_2$ and $E(Z^3) = B_1 + B_2$. Compute $E(Z^4)$ and show, in particular, that $E(Z^4) \neq C_1 + C_2$.

(a) The rv $(X - E(X))^2$ is non-negative and is not identically zero because X is not constant. Therefore

$$0 < E((X - E(X))^2) = E(X^2 - 2E(X)X + E(X)^2) =$$
$$E(X^2) - 2E(X)E(X) + E(X)^2 = E(X^2) - E(X)^2.$$

So $E(X^2) > E(X)^2$.

(b)By the binomial theorem, or direct multiplication:

$$E(Z^2) = E(X_1^2 + 2X_1X_2 + X_2^2) = A_1 + 2E(X_1X_2) + A_2,$$
$$E(Z^3) = E(X_1^2 + 3X_1^2X_2 + 3X_1X_2^2 + X_2^3) = B_1 + 3E(X_1^2X_2) + 3E(X_1X_2^2) + B_2,$$
$$E(Z^4) = E(X_1^4 + 4X_1^3X_2 + 6X_1^2X_2^2 + 4X_1X_2^3 + X_2^4)$$
$$= C_1 + 4E(X_1^3X_2) + 6E(X_1^2X_2^2) + 4E(X_1X_2^3) + C_2.$$

By independence,

$$E(X_1X_2) = E(X_1)E(X_2) = 0, \quad E(X_1^2X_2) = E(X_1^2)E(X_2) = 0,$$
$$E(X_1X_2^2) = E(X_1)E(X_2^2) = 0,$$
$$E(X_1^3X_2) = E(X_1^3)E(X_2) = 0, \quad E(X_1X_2^3) = E(X_1)E(X_2^3) = 0,$$
$$E(X_1^2X_2^2) = E(X_1^2)E(X_2^2) = A_1A_2.$$

Because the rv's are nonconstant, $A_1, A_2 > 0$. Therefore,

$$E(Z^2) = A_1 + A_2, \quad E(Z^3) = B_1 + B_2,$$
$$E(Z^4) = C_1 + 6A_1A_2 + C_2 > C_1 + C_2.$$

2-(18 points) Let $X \sim Poiss(\lambda)$ so that the pmf of X is $f_X(n) = \frac{\lambda^n}{n!} \cdot e^{-\lambda}$.

(a) Compute the MGF $E(e^{tX})$ and use it to compute $E(X)$ and $Var(X)$.

(b) Let X_1 and X_2 have Poisson distributions each with mean λ.

(i) Show that if X_1 and X_2 are independent, then $X = X_1 + X_2$ has a Poisson distribution. Explain where independence is used.

(ii) Show that if $X_1 = X_2$ then $X = X_1 + X_2$ does not have a Poisson distribution. Compute $P(X = 1)$ and $P(X = 2)$.

(a) $f_X(n) = e^{-\lambda}\frac{\lambda^n}{n!}$. So

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}.$$

So $M_X(t) = exp(\lambda(e^t - 1))$.

(b) (i) If $X_1 \sim Poiss(\lambda_1), X_2 \sim Poiss(\lambda_2)$ then $M_X(t) = E(e^{tX_1 + tX_2}) = E(e^{tX_1} e^{tX_2})$. For independent rv's the expectation of the product is the product of the expectations and so

$$M_X(t) = M_{X_1}(t) M_{X_2}(t) = exp((\lambda_1 + \lambda_2)(e^t - 1)).$$

Because the MGF characterizes the distribution, $X \sim Poiss(\lambda_1 + \lambda_2)$.

Alternatively, $P(X = n) = \sum_{k=0}^{n} P(X_1 = k \& X_2 = n - k)$. By independence this equals

$$\sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_1^{n-k}}{(n-k)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}.$$

By the Binomial Theorem this equals $e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}$. So $X \sim Poiss(\lambda_1 + \lambda_2)$.

If $\lambda_1 = \lambda_2 = m$ then $X \sim Poiss(2\lambda)$.

(ii) If $X_1 = X_2$ then $X = 2X_1$ has range only the even nonnegative integers and so is not Poisson. In particular, $P(X = 1) = 0$ and $P(X = 2) = P(X_1 = 1) = \lambda e^{-\lambda}$.

3-(20 points) Let X, Y be rv's with joint density

$$f_{X,Y}(x,y) = C(x+y)dxdy \quad \text{for} \quad 0 < x, y < 1 \quad \text{and} \quad = 0 \quad \text{otherwise.}$$

(i) Compute the constant C so that this defines a joint pdf.

(ii) Compute the marginal distributions for X and Y and determine whether X and Y are independent (justifying your answer).

(iii) Compute the conditional pdf of Y given X, i.e. $f_{Y|X}(y|x)\ dy$.

(iv) For $Z = XY$, compute the survival function G_Z and the pdf of Z.

(v) Compute the expectations of X, Y and Z.

(i), (ii) $f_X(x) = (\int_0^1 f_{X,Y}(x,y)dy)\ dx = (\int_0^1 x + ydy)\ dx = (x + \frac{1}{2})\ dx$ for $0 < x < 1$.

$$\int_0^1 \int_0^1 x + ydydx = \int_0^1 x + \frac{1}{2}dx = 1$$

and so $f_{X,Y}$ is a pdf with $C = 1$.

By symmetry, $Y \sim X$ and $f_Y(y) = y + \frac{1}{2}$.

They are not independent because the joint density is not the product of the marginals.

(iii) $f_{Y|X}(y|x)\ dy = [f_{X,Y}(x,y)\ dydx] \div (f_X(x)\ dx) = \frac{x+y}{x+1/2}\ dy$, for $0 < x, y < 1$.

(iv) For $0 < z < 1$,

$$G_Z(z) = P(z \leq Z) = \int_{x=z}^{x=1} \int_{y=z/x}^{y=1} (x+y)dydx =$$

$$\int_{x=z}^{x=1} x + \frac{1}{2} - z - \frac{z^2}{2x^2}dx = 1 - 2z + z^2 = (1-z)^2.$$

Hence, the pdf $f_Z(z) = -G'_Z(z) = 2(1-z)$.

Alternatively, but harder, one can transform, mapping $(x,y) \to (w,z)$ with $w = x, z = xy$. This maps the square with $0 < x, y < 1$ to the triangle $0 < z < w < 1$. The inverse map is $(w,z) \to (x,y)$ with $x = w, y = z/w$.

The Jacobian matrix is given by

$$\frac{\partial(x,y)}{\partial(w,z)} = \begin{pmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -z/w^2 & 1/w \end{pmatrix}.$$

So the Jacobian factor $|\frac{\partial(x,y)}{\partial(w,z)}| = 1/w$.

$$
\begin{aligned}
f_{W,Z}(w,z)dwdz &= f_{X,Y}(x(w,z),y(w,z))\left|\frac{\partial(x,y)}{\partial(w,z)}\right|dwdz \\
&= (w+z/w)\cdot(1/w)dwdz = (1+z/w^2)dwdz
\end{aligned}
$$

$$
f_Z(z)dz = \left(\int_z^1 (1+z/w^2)dw\right)dz = (w-z/w)|_z^1 dz = 1-z-z+1 = 2(1-z).
$$

(v)

$$
\begin{aligned}
E(X) &= \int_0^1 x(x+\frac{1}{2})dx = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}. \\
E(Z) &= \int_0^1 z2(1-z)dz = 1-\frac{2}{3} = \frac{1}{3}.
\end{aligned}
$$

Since $X \sim Y$, $E(Y) = E(X)$.

4-(14 points) Assume that $X_1, \ldots, X_5$ are independent rv's each with distribution $Unif(0,1)$. Let $Y_1, \ldots, Y_5$ be the order statistics.

(i) Compute the densities $f_{Y_5}(y_5)dy_5$, $f_{Y_4}(y_4)dy_4$, and $f_{Y_4Y_5}(y_4,y_5)dy_4dy_5$,.

(ii) Compute the expectations $E(Y_5)$ and $E(Y_4)$.

The pdf is $f(x)dx = 1dx$ and the cdf is $F(x) = x$ for $0 < x < 1$.

(i)

$$
\begin{aligned}
f_{Y_5}(y_5)dy_5 &= 5f(y_5)F(y_5)^4dy_5 = 5y_5^4dy_5, \\
f_{Y_4}(y_4)dy_4 &= 5\cdot 4f(y_4)F(y_4)^3(1-F(y_4)) = 20y_4^3(1-y_4)dy_4. \\
f_{Y_4Y_5}(y_4,y_5)dy_4dy_5 &= 5\cdot 4f(y_4)f(y_5)F(y_4)^3dy_4dy_5 = 20y_4^3dy_4dy_5 \quad \text{for } 0 < y_4 < y_5 < 1.
\end{aligned}
$$

(ii)

$$
E(Y_5) = \int_0^1 5y_5^5dy_5 = \frac{5}{6}, \quad E(Y_4) = \int_0^1 20y_4^4(1-y_4)dy_4 = \frac{4}{6}.
$$

5-(18 points) $X_1, \ldots, X_n$ is a sample with distribution with exponential distribution $Exp(\theta)$ so that $f_X(x)\ dx = \theta e^{-\theta x}\ dx$, and it is known that $\theta \le 1$. Hence, the range of the parameter we wish to estimate is $0 < \theta \le 1$. Compute the MLE for θ.

$f_X(x) = \theta e^{-\theta x}$ and so $L(\theta) = \theta^n exp(-\theta \sum_{i=1}^n \ x_i)$ and

$$\ell(\theta) = n\ln(\theta) - \theta \sum_{i=1}^{n} \ x_i$$

$$\frac{\partial \ell}{\partial \theta} = n\theta^{-1} - \sum_{i=1}^{n} \ x_i$$

So $\frac{\partial \ell}{\partial \theta} = 0$ when $\theta = n \div (\sum_{i=1}^n \ x_i)$ with $\frac{\partial^2 \ell}{\partial \theta^2} = -n\theta^{-2} < 0$.

However, θ is restricted to interval $(0, 1]$. If $n \div (\sum_{i=1}^n \ x_i) \leq 1$ then this is the point at which maximum ℓ is maximum. If $n \div (\sum_{i=1}^n \ x_i) > 1$ then $\frac{\partial \ell}{\partial \theta} > 0$ and the maximum value occurs occurs at the right endpoint $\theta = 1$. Thus, the MLE is

$$\hat{\theta} = minimum(1, 1/\bar{X}).$$

6- (18 points) $X_1, \ldots, X_n$ is a sample with distribution $Unif(0, \theta)$.

(i) Compute the MLE $\hat{\theta}$ for θ and the MLE for the mean.

(ii) Compute the cdf, pdf and mean of $M = \hat{\theta}$. Show that $\hat{\theta}$ is a biased estimate of θ.

(iii) For $M = \hat{\theta}$, compute the range and the cdf of $Z = \frac{M}{\theta}$ and show that this is a known distribution, i.e. it does not depend on θ. Use it as a pivot variable to obtain a 90% confidence interval for θ (Hint: Compute q so that $P(Z \leq q) = .1$)

(i) $L(\theta) = (1/\theta^n)$ for $0 \le x_1, \ldots, x_n \le \theta$ and $= 0$ otherwise. Since $(1/\theta^n)$ is decreasing in θ, the MLE $\hat{\theta} = M = \max(X_1, \ldots, X_n)$.

The mean value of X is $\theta/2$ and so the MLE for the mean is $M/2$ and not $\bar{X}$.

(ii) The cdf is $F_X(t) = \frac{t}{\theta}, \quad 0 < t < \theta$. Since $M \le t$ if and only if $X_1, \ldots, X_n \le t$, the cdf of M is $F_M(t) = (\frac{t}{\theta})^n, \quad 0 < t < \theta$ and so the pdf of M is $f_M(t) = F'_M(t) = n\frac{t^{n-1}}{\theta^n}$ and so

$$E(M) = \int_0^\theta n\frac{t^n}{\theta^n}dt = \frac{n}{n+1}\theta.$$

So the estimate is biased.

(iii) The range of M is $(0, \theta)$ and so the range of Z is $(0, 1)$. For $t \in (0, 1)$, $F_Z(t) = F_M(t\theta) = F_X(t\theta)^n$. $F_X(x) = x/\theta$ for $0 < x < \theta$. So $F_Z(t) = t^n$. With $q = .1^{1/n}$, $P(Z \le q) = .1$ and so $P(Z > q) = .9$. That is, $P(\frac{M}{q} > \theta) = .9$. On the other hand, $P(\theta > M) = 1$. Thus, the 90% confidence interval is

$$(\max(x_1, \ldots, x_n), \max(x_1, \ldots, x_n)/.1^{1/n}).$$

Notice that as n tends to infinity, $.1^{1/n} < 1$ increases with limit 1, and so $1/.1^{1/n} > 1$ decreases with limit 1.