Math 375 Fall 2012 : Final Exam, December 19, 2012

Name:

Score:

1	2	3	4	5	6	7	8	extra	total

Complete **6** *of the following 8 questions.* **Mark clearly which of problems 1-8 is** *not* **to be graded. Your grade will be computed from your 5 highest-scoring answers.** *Please write clearly as well. It is not strictly necessary to simplify factorials and combinations. You can use a calculator (which is not necessary), but no phones or devices that connect to the internet.* There is an extra credit problem at the end.

1. Six students sit in a row of 6 seats. 3 of the students are girls and 3 of the students are boys.

   (a) (8 pts) In how many different ways can they sit in a row?

   (b) (4 pts) In how many ways if the boys sit together and the girls sit together?

   (c) (4 pts) In how many ways if only the boys must sit together?

   (d) (4 pts) In how many ways if no two people of the same sex are allowed to sit together?

2. Describe a random variable X and compute its expectation and variance and cumulative distribution function. X should not be a constant. Be sure to include a specification of

   (a) (3 pts) The sample space

   (b) (3 pts) The probability assignment

   (c) (2 pts) The explanation that the probability assignment is valid

   (d) (4 pts) The definition of the random variable X

   (e) (4 pts) The expectation and variance of X

   (f) (4 pts) The cumulative distribution function (cdf) of X.

3. The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad \text{where } 0 < x < 1 \text{ and } 0 < y < 2 \tag{1}$$

(a) (4 pts) Verify that this is indeed a joint density function.

(b) (4 pts) Find $P(X > Y)$.

(c) (3 pts) Compute the density function (pdf) of X.

(d) (3 pts) Find $P\left(Y > \frac{1}{2} \,\middle|\, X < \frac{1}{2}\right)$.

(e) (3 pts) What is the formula for $E[X]$? (You can express it as an integral; you don't need to simplify or compute it.)

(f) (3 pts) What is the formula for $E[X^2]$? (You can express this as an integral as well.)

4. A laboratory blood test is 95% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a “false positive” result for 1% of the healthy persons tested. (That is, if a healthy person is tested, then, with probability 0.01, the test result will imply that he or she has the disease.) 0.5% of the population actually has the disease.

   (a) (13 pts) What is the probability that a patient has the disease given that the test result is positive?

   (b) (7 pts) Suppose a patient takes the test twice. What is the probability that a patient has the disease given that both test results are positive? What independence assumptions are you making?

5. Give an example of a probability space with two random variables X and Y such that $Var[X+Y] \neq Var[X] + Var[Y]$.

   (a) (10 pts) Give the example and show $Var[X + Y] \neq Var[X] + Var[Y]$.

   (b) (5 pts) Are X and Y independent? Justify your answer using the definition of independence for two random variables.

   (c) (5 pts) In this case, what is $E[X]$ and $E[X + Y]$?

6. Let X be normally distributed with parameters μ and σ^2.

   (a) (5 pts) Is X continuous? Why?

   (b) (4 pts) Give an expression for the cumulative distribution function (cdf) of X.

   (c) (4 pts) What is $P(X = 3)$?

   (d) (4 pts) Let $Y = 3X + 2$. Compute $E[Y]$ and $Var[Y]$.

   (e) (3 pts) Show that $P(\mu - 1 < X < \mu) = P(\mu < X < \mu + 1)$.

7. Suppose, over the course of a semester, Barack's teacher Mr. Killjoy assigns 60 homework problems in 10 batches of 6 problems each. After each batch is assigned, Mr. Killjoy gives a quiz consisting of a question randomly chosen from the batch. Barack, a student in the class, decides he needs to conserve his energy and decides to work on only a portion of the problems. He will do k problems in each of the 10 batches, but he needs to choose the number k. The problems are hard but Barack has a good memory, so that Barack assumes he will get a 0 if he hasn't completed the problem on the quiz, whereas he will get a 100% if he has completed it. He decides he will be satisfied if he gets at least an 80% average on the homework portion of the class grade. How should Barack choose k?

   (a) (6 pts) Let X be the number of quizzes on which Barack gets a 100. Let Y be Barack's grade on the homework portion of the test. Write an expression for Y written in terms of X.

   (b) (8 pts) What is the probability mass function for X? (Note this will depend on the unknown k).

   (c) (5 pts) Write an expression for the inequality that must be satisfied (involving k), so that Barack has at least a 75% chance of satisfying his desired goal of 80% for the homework grade.

   (d) (1 pt) What must Barack do to guarantee (that is, with 100% confidence) that he receives at least 80% on his total homework grade?

8. Suppose that a fair die is rolled 100 times. Let X_i be the value obtained on the i'th roll.

   (a) (5 pts) What is $P\left(\sum_{i=1}^{100} X_i = 100\right)$?

   (b) (3 pts) What is $P\left(\sum_{i=1}^{100} X_i = 101\right)$?

   (c) (2 pts) What is $E\left[\sum_{i=1}^{100} X_i\right]$?

   (d) (4 pts) Give an expression for an approximation of $P\left(\sum_{i=1}^{100} X_i \leq 380\right)$ based on $\Phi(t)$, the cumulative distribution function (cdf) of a standard normal random variable.

   (e) (3 pts) Let $W = \prod_{i=1}^{100} X_i$. What is $E[W]$? Justify your answer.

   (f) (3 pts) Give an expression for an approximation of $P\left(W \leq a^{100}\right)$, where $1 < a < 6$, (using the central limit theorem). *(Hint: consider taking the logarithm of W.)*

9. **Extra credit (10 pts): Explain the Monty Hall problem.** Recall the Monty Hall problem discussed in the first day of class. Monty hall is the host of a game show, in which there are three closed doors, with a prize hidden behind one of the doors. A contestant first chooses one of the doors. Monty Hall proceeds to open one of the *other* two doors. Monty Hall purposely opens an empty door (with no prize behind it). So now there are two closed doors, and the contestant must open one of them. If she opens the one with the prize behind it, she wins the prize.

   So the contestant has two choices. She can either *STAY,* meaning that she opens the door that she originally chose, or she can *SWITCH,* meaning that she opens the other door, which she did not originally choose. Explain why the contestant should switch.

   Credit will be given for sound and clear explanations.