MATH 375 Final Exam Fall, 2018

Part I: Do both problems.

1. Consider the seven letter word MERRIER.
(a) How many distinct "words" can be formed by rearranging the letters?
(b) Two letters are chosen with replacement. What is the probability that they are both R's?
(c) Two letters are chosen without replacement. What is the probability that they are both the same (that is, both M's or both E's etc)?

2. Assume that among 10 coins, 1 is double-headed and the rest are fair two-sided coins. A coin is chosen at random and flipped 6 times. Let X be the number of heads which occur. Let D be the indicator function with $D = 1$ if the coin is double-headed and $= 0$ otherwise.
(a) Compute the expected value $E(X)$. (Hint: First compute the conditional expected value $E(X|D)$.)
(b) Assuming $X = 6$, i.e. all of the flips are heads, what is the probability that $D = 1$, i.e. that the coin is double-headed?

Part II: Do six (6) out of the following eight (8) problems.

3. Give a story proof that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. (Hint: consider a committee of size k chosen from n people, one of whom is named Ralph).

4. Among the math professors 60% are hard graders. If you have a hard grader, then the probability of your getting an A is .10. Otherwise, your probability of getting an A is .30. Congratulations, you got an A. What is the probability that you had a hard grader?

5. Suppose that $X \sim Geom(p)$. What is X counting? Describe the mass function, the PMF, of X and compute the moment generating function, the MGF, of X and the expected value of X.

6. (a) If $X \sim Bin(n, p)$, then how can X be expressed using Bernoulli random variables?

(b) Assuming that $X, Y \sim Bin(n, p)$ and that X and Y are independent, show that $X + Y \sim Bin(2n, p)$. Explain why if $X = Y$, then $X + Y$ cannot be a binomial random variable (Hint: Consider the values.)

7. (a) For $X \sim Pois(\lambda)$ describe the PMF, of X and compute the MGF of X.

(b) Show that if $X \sim Pois(\lambda), Y \sim Pois(\mu)$ and X and Y are independent, then $X + Y$ is Poisson as well. Explain where independence is used.

8. Assume that $X \sim Exp(\lambda)$.

(a) Compute $P(1 < X < 10)$ and the survival function G_X with $G_X(t) = P(X > t)$.

(b) Show that $P(X > t + s | X > s) = P(X > t)$.

9. Assume that $f_X(x) = C \cdot x^3$ for $0 < x < 1$ and $= 0$ otherwise.

(a) With what value must C have so that f_X is a density function, a PDF? If X has this density function, what are $E(X)$ and $Var(X)$?

(b) If $X_1, \ldots, X_n$ are i.i.d. random variables, each with density function f_X, and $M = \max(X_1, \ldots, X_n)$, what is $P(M < \frac{1}{2})$?

10. (a) For an event A define what it means for I_A to be the indicator random variable of A.

(b) For independent events A, B express $I_{A\cap B}$ in terms of I_A and I_B.

(c) For mutually exclusive events A, B compute $I_{A\cap B}$.

Part III: Do two (2) out of the following three (3) problems.

11. The number N of eggs that a chicken lays is Poisson distributed with $N \sim Pois(\lambda)$. Each egg hatches with probability p and so does not hatch with probability $q = 1 - p$. The different eggs are independent. Let X be the number which hatch and Y be the number which do not so that $X + Y = N$.

(a) What is the PMF of the conditional distribution of X assuming $N = n$?

(b) Compute the joint PMF $f_{XY}(i, j) = P(X = i, Y = j)$, for $i, j = 0, 1, \ldots$. (Hint: Observe that $P(P(X = i, Y = j) = P(X = i, N = i + j)$.)

(c) Explain why X and Y are independent Poisson random variables.

12. A piece of random length X is broken from a stick of length 1 so that $X \sim Unif(0,1)$. The a piece of random length Y is broken from the first piece.
(a) Compute the conditional density of Y assuming $X = x$.
(b) Compute the conditional expectation $E(Y|X)$ and the expectation $E(Y)$.

13. Assume that the pair of random variables (X, Y) has joint PDF

$$f_{X,Y}(x,y)\ dydx \ = \ 8xy\ dydx$$

when $0 < x < y < 1$ and $= 0$ otherwise.
(a) Compute the density functions f_X and f_Y and the conditional density function $f_{Y|X}$.
(b) Compute the expectations $E(X)$, $E(Y)$ and the covariance $Cov(X,Y)$.