MATH 346 Test 2 Solutions Fall, 2022

1. (16 points) Determine whether each of the following is or is not a subspace of the vector space $M_{2\times 2}$ of 2×2 matrices. That is, if it is a subspace, check that the required conditions hold. If it is not give an example to show that some condition fails. (a) Matrices with every element greater or equal to 0. That is, all $A_{ij} \geq 0$.

(b) Non-invertible matrices. That is, A^{-1} does not exist.

(c) Symmetric matrices. That is, $A^T = A$.

(a) Not a subspace. $(-1)\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ is not in the set.

(b) Not a subspace. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not in the set.

(c) Subspace. If $A^T = A$ and $B^T = B$, then $(A+B)^T = A^T + B^T = A + B$ and $(cA)^T = c(A^T) = cA$.

2. (18 points) For an $m \times n$ matrix A, the null space, $Null(A) = \{X \in \mathbb{R}^n : AX = \theta\}$ where θ is the 0 matrix. That is, $Null(A)$ is the solution space of the associated homogeneous system.

(a) Show that $Null(A)$ is a subspace of R^n.

(b) For the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$ find a basis for $Null(A)$.

(a) If $AX = \theta$ and $AY = \theta$ then $A(X+Y) = AX + AY = \theta + \theta = \theta$ and $A(cX) = c(AX) = c\theta = \theta$. So the nullspace is a subspace.

(b) We put A is Reduced Row Echelon Form.

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 4 & 4 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$$

$R_2 \to \frac{1}{2}R_2$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$$

$R_2 \to R_2 - R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_3, R_4 \to R_4 - R_3.$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 0 \end{pmatrix}$$

$R_4 \to \frac{1}{2}R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$R_2 \leftrightarrow R_4$

$$Q = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution of the homogeneous system is : $x_5 = 0, x_4 = r, x_3 = -r, x_2 = s, x_1 = -3s - r$. The basis vectors are $e_r = (-1, 0, -1, 1, 0), e_s = (-3, 1, 0, 0, 0)$.

3. (18 points) Assume that $\{v_1, \dots, v_n\}$ is a list of vectors in a vector space V.

(a) Define what it means that the list $\{v_1, \dots, v_n\}$ is linearly independent.

(b) Show that if v_n is a linear combination of $\{v_1, \dots, v_{n-1}\}$, then the list $\{v_1, \dots, v_n\}$ is linearly dependent, i.e. not linearly independent.

(c) Show that if $\{v_1, \dots, v_n\}$ is linearly independent, then $\{v_1, \dots, v_{n-1}\}$ does not span. V.

(a) $\{v_1, \dots, v_n\}$ is li when $c_1v_1 + \cdots + c_nv_n = \theta$ only when $c_1 = \cdots = c_n = 0$.

(b) If $v_n = x_1v_1 + \dots x_{n-1}v_{n-1}$, then $\theta = x_1v_1 + \dots x_{n-1}v_{n-1} + (-1)v_n$.

(c) If $\{v_1, \dots, v_{n-1}\}$ spans then v_n is a linear combination of $\{v_1, \dots, v_{n-1}\}$, and so the list $\{v_1, \dots, v_n\}$ is linearly dependent as in (b).

4. (16 points) We use the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$ from problem 2.

(a) Find a basis for the column space of A and show that $B = \begin{pmatrix} 1 & 0 & -1 & 1 \end{pmatrix}^T$ is a linear combination of the columns of A.

(b) Find a basis for the row space of A.

(a) With A and Q as in problem 2, the leading ones in Q are in columns 1, 3 and 5. So columns 1, 3 and 5 of A provide a basis for the column space.

To obtain B are a linear combination of the columns of A, we put the augmented matrix $(A|B)$ is Reduced Row Echelon Form.

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix} \text{ find a basis for } Null(A).$$

(a) If $AX = \theta$ and $AY = \theta$ then $A(X+Y) = AX + AY = \theta + \theta = \theta$ and $A(cX) = c(AX) = c\theta = \theta$. So the nullspace is a subspace.

(b) We put A is Reduced Row Echelon Form.

$$(A|B) = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & | & 1 \\ 2 & 6 & 4 & 6 & 6 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & | & -1 \\ 0 & 0 & 2 & 2 & 1 & | & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 4 & 4 & 6| & -2 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to \frac{1}{2}R_2$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 2 & 2 & 3| & -1 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to R_2 - R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 2| & -2 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_3, R_4 \to R_4 - R_3.$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 0| & 0 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 0| & 2 \end{pmatrix}$$

$R_4 \to \frac{1}{2}R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 0| & 0 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 1 & 1 & 0| & 1 \end{pmatrix}$$

$R_2 \leftrightarrow R_4$

$$Q = \begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 1 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 0 & 0 & 0| & 0 \end{pmatrix}$$

The system is consistent and the general solution is given by $x_5 = -1, x_4 = r, x_3 = 1 - r, x_2 = s, x_1 = 1 - 3s - r$. If we set $r = s = 0$ we may use $(x_1, x_2, x_3, x_4, x_5) = (1, 0, 1, 0, -1)$.

(b) The three nonzero rows of Q provide a basis for the row space.

5. (16 points) Assume that A is an $m \times n$ matrix.
(a) Assume that the columns of A form a linearly independent list. What is the rank of A? (Explain)
If, in addition, $m = n$ so that the matrix is square, does this imply that A has an inverse? (Explain)
(b) If $m < n$, can the columns form a linearly independent list? (Explain)

(a) If A is row equivalent to Q with Q in Reduced Echelon Form, then linear independence of the columns requires that there be a leading one in every column and so the rank is n.
If $m = n$, then there is a leading one in every column only when $Q = I_n$. This implies that A has an inverse.
(b) Since there is at most one leading one in every row, the rank is at most m. So if $m < n$ then the rank is less than n and so the columns cannot be linearly independent.

6. (16 points)(a) Using the definition of dimension, show that the dimension of $\mathbb{R}^3$ is 3.
(b) Show that the map $det : M_{2\times 2} \to \mathbb{R}$, associating to each matrix A its determinant, is not a linear map.

(a)The columns of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ provide a basis for R^3 consisting of three vectors and so the dimension is 3.
(b) $det(A+B)$ is usually not equal to $det(A)+det(B)$ and $det(cA) = c^2 det(A)$ not $cdet(A)$.