1. (16 points) Determine whether each of the following is or is not a subspace of the vector space $M_{2\times 2}$ of 2×2 matrices. That is, if it is a subspace, check that the required conditions hold. If it is not give an example to show that some condition fails.

(a) Nonnegative matrices. That is, the set $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \geq 0\}$

(b) Non-invertible matrices. That is, the set $\{A : A^{-1}$ does not exist $\}$.

(c) Symmetric matrices. That is, $\{A : A^T = A\}$.

(b) Not a subspace. $(-1)\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ is not in the set.

(b) Not a subspace. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not in the set.

(c) Subspace. If $A^T = A$ and $B^T = B$, then $(A+B)^T = A^T + B^T = A + B$ and $(cA)^T = c(A^T) = cA$.

2. (16 points) We use the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$.

(a) Show that $B = \begin{pmatrix} 1 & 0 & -1 & 1 \end{pmatrix}^T$ is a linear combination of the columns of A and find a basis for the column space of A

(b) Find a basis for the row space of A.

(a) To obtain B are a linear combination of the columns of A, we put the augmented matrix $(A|B)$ is Reduced Row Echelon Form.

$$(A|B) = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & | & 1 \\ 2 & 6 & 4 & 6 & 6 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & | & -1 \\ 0 & 0 & 2 & 2 & 1 & | & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 4 & 4 & 6| & -2 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to \frac{1}{2}R_2$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 2 & 2 & 3| & -1 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to R_2 - R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 2| & -2 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_3, R_4 \to R_4 - R_3$.

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 0| & 0 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 0| & 2 \end{pmatrix}$$

$R_4 \to \frac{1}{2}R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 0| & 0 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 1 & 1 & 0| & 1 \end{pmatrix}$$

$R_2 \leftrightarrow R_4$

$$(Q|\tilde{B}) = \begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 1 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 0 & 0 & 0| & 0 \end{pmatrix}$$

The system is consistent and the general solution is given by $x_5 = -1, x_4 = r, x_3 = 1 - r, x_2 = s, x_1 = 1 - 3s - r$. So B is a linear combination of the columns of A.

For A put in reduced echelon form, obtaining Q, there are leading ones in the first, third and fifth columns. So the basis is given by the first, third and fifth columns of the original matrix A

(b) The three nonzero rows of Q provide a basis for the row space.

3. (24 points) Assume that $\{v_1, \ldots, v_n\}$ is a list of vectors in a vector space V.

(a) Define what it means that the list $\{v_1, \ldots, v_n\}$ is linearly independent.

(b) Show that if v_n is a linear combination of $\{v_1, \ldots, v_{n-1}\}$, then the list $\{v_1, \ldots, v_n\}$ is linearly dependent, i.e. not linearly independent.

(c) Show that if $\{v_1, \ldots, v_{n-1}\}$ spans V, then $\{v_1, \ldots, v_{n-1}, v_n\}$ is linearly dependent.

(d) Show that a list of two vectors $\{v_1, v_2\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

(a) $\{v_1, \ldots, v_n\}$ is li when $c_1 v_1 + \cdots + c_n v_n = \theta$ only when $c_1 = \cdots = c_n = 0$.

(b) If $v_n = x_1 v_1 + \ldots x_{n-1} v_{n-1}$, then $\theta = x_1 v_1 + \ldots x_{n-1} v_{n-1} + (-1) v_n$. The coefficient of v_n is not zero.

(c) If $\{v_1, \ldots, v_{n-1}\}$ spans V, then v_n is a linear combination of $\{v_1, \ldots, v_{n-1}\}$, and so the list $\{v_1, \ldots, v_n\}$ is linearly dependent as in (b).

(d) If $v_2 = Cv_1$, then as in (b) $\theta = Cv_1 + (-1)v_2$. If $\theta = c_1v_1 + c_2v_2$ and $c_2 \neq 0$, then $v_2 = Cv_1$ with $C = -c_1/c_2$.

4. (16 points) For an $m \times n$ matrix A, the null space, $Null(A) = \{X \in \mathbb{R}^n : AX = \theta\}$ where θ is the 0 matrix. That is, $Null(A)$ is the solution space of the associated homogeneous system.

(a) Show that $Null(A)$ is a subspace of R^n.

(b) For the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$ given in problem 2, find a basis for $Null(A)$. (You may use your work from problem 2.)

(a) If $AX = \theta$ and $AY = \theta$ then $A(X+Y) = AX + AY = \theta + \theta = \theta$ and $A(cX) = c(AX) = c\theta = \theta$. So the nullspace is a subspace.

(b) Using our work from (1) We put A is Reduced Row Echelon Form to obtain $Q = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution of the homogeneous system is : $x_5 = 0, x_4 = r, x_3 = -r, x_2 = s, x_1 = -3s - r$. The basis vectors are $e_r = (-1, 0, -1, 1, 0), e_s = (-3, 1, 0, 0, 0)$.

5. (16 points) Consider the following list of polynomials in $\mathcal{P}_3$, the vector space of polynomials of degree at most three.

$$S = \{t^3 + 2t^2, 3t^3 + 6t^2, 4t^2 + 2, t^3 + 6t^2 + 2, 6t^2 + t + 1\}.$$

(a) Show that the list S is linearly dependent.

(b) Obtain a basis for the $Span(S)$ and compute the dimension of $Span(S)$.

(a) Observe that $3t^3 + 6t^2 = 3(t^3 + 2t^2$ so the list is ld. Alternatively, proceed to (b)

(b) Translating the polynomials into column vectors we obtain the matrix A given in problems 2 and 4. In reduced row echelon form we obtain as before

$$Q = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is not a leading one in every column, the list of columns is linearly dependent. Also we saw that a basis for the column space consisted of the first, third and fifth columns of A. Translating back to polynomials, a basis for $Span(S)$ is

$$\{t^3 + 2t^2, 4t^2 + 2, 6t^2 + t + 1\}.$$

6. (12 points)(a) For a vector space V, define "the dimension of V".
(b) Using the definition of dimension, show that the dimension of $\mathbb{R}^3$ is 3.

(a) The dimension of V is the number of vectors in any basis.

(b)The columns of $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ provide a basis for $\mathbb{R}^3$ consisting of three vectors and so the dimension is 3.