MATH 346 Test 1 Solutions Spring, 2025

1. (12 points) If $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 3 & 0 \\ 1 & 1 \end{pmatrix}$, compute each of the following if it can be done. If it cannot be done, If it cannot be done, explain why not.

$$(i)\ \ AB \qquad (ii)\ \ AB^T \qquad (iii)\ \ A+B \qquad (iv)\ \ A-A^T.$$

(i) can't multiply a 2×2 by a 3×2.

(ii) $AB^T = \begin{pmatrix} 14 & 9 & 2 \\ 11 & 6 & 3 \end{pmatrix}$.

(iii) can't add with different shapes.

(iv)$A - A^T = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$

2. (15 points) (a) If A is an $n \times n$ matrix define what it means for B to be *inverse* of A.

(b) If A is an $n \times n$ matrix with a row of zeroes, show - without using determinants - that A is not *invertible*. (That is, it does not have an inverse).

(c) If A and B are invertible $n \times n$ matrices, show that the product AB is invertible and compute its inverse.

(a) B is the inverse of A when it cancels A. That is, $AB = I_n = BA$.

(b) If A is an $n \times n$ matrix with a row of zeroes, show that A is not invertible.

If row i of A is a row of zeroes then for any $n \times n$ matrix B, row i of AB is a row of zeroes and so $AB \neq I_n$.

(c) If A and B are invertible $n \times n$ matrices, show that the product AB is invertible and compute its inverse.

$(AB)(B^{-1}A^{-1}) = I_n$ and $(B^{-1}A^{-1})(AB) = I_n$. Since $(B^{-1}A^{-1})$ cancels AB it is the inverse of AB.

3. (14 points) (a) For the row operation: To Row 2 add a copy of -5 times Row 3, write down the associated 3×3 elementary matrix E and its inverse E^{-1}.

(b) For the following system compute the inverse of the coefficient matrix and use it to solve the system.

$$\begin{array}{rcrcr} 3x & + & 2y & = & 8 \\ x & + & 2y & = & -12 \end{array}$$

(a) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$

(b) The determinant of $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ is $6-2 = 4$ and so the inverse is $\frac{1}{4}\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$

So $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ -12 \end{pmatrix} = \begin{pmatrix} 10 \\ -11 \end{pmatrix}$.

4. (14 points)(a) For A a 7×5 matrix in echelon form, what is the largest rank A can have? What can you say about the bottom row? (Explain)

The rank is at most 5 since every column has at most one leading 1. So there are at least two rows without leading 1's and these bottom two rows are rows of zeroes.

(b) When can a 3×3 matrix B have rank zero? (Explain)

If a matrix in echelon form has no leading 1's then every row is a zero row and so the matrix equal 0_3. Since B is row equivalent to the zero matrix it must be the zero matrix.

(c) If C a 5×5 matrix with $det(C) = 3$, what is $det(2C)$?

$det(2C) = 2^5 det(C) = 32 \cdot 3 = 96$.

5. (14 points) Assume that A is a 5×5 matrix with rank 5.

(a) If Q is in reduced row echelon form and is row equivalent to A what does Q have to be?

(b) Explain how you know that A has an inverse. You may use the fact that elementary matrices are invertible.

(a) $Q = I_5$.

(b) Q is obtained from A by a sequence of row operations $Q = Op_k(...(Op_1(A)...)$ and this is the same as multiplying by the corresponding elementary matrices. $I_5 = Q = (E_k...E_1)A$. So $U = E_k...E_1$ cancels A and $A = U^{-1}$.

6. (16 points)(a) Put the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & -5 \\ 1 & 0 & -1 & 1 & 2 \\ 1 & 0 & -1 & 0 & -2 \\ 2 & 1 & 1 & -1 & -7 \end{pmatrix}$$

in *reduced echelon form* and compute its *rank.*

$$R_1 \leftrightarrow R_2 : \begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 3 & 1 & 0 & 0 & -5 \\ 1 & 0 & -1 & 0 & -2 \\ 2 & 1 & 1 & -1 & -7 \end{pmatrix}$$

$$R_2 \to R_2 - 3R1, R_3 \to R_3 - R_1, R_4 \to R_4 - 2R_1 : \begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 3 & -3 & -11 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 1 & 3 & -3 & -11 \end{pmatrix}$$

$$R_4 \to R_4 - R_2, R_3 \to (-1)R_3 : \begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 3 & -3 & -11 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_2 \to R_2 + 3R_3, R_1 \to R_1 - R_3 : \begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank = 3.

(b) Compute the general solution for the system of equations which has A as its augmented matrix.
$x_4 = 4, \quad x_3 = r, \quad x_2 = 1 - 3r, \quad x_1 = -2 + r.$

7. (14 points) Compute the inverse and the determinant of the following matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$(A|I_4) = \begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_2 \leftrightarrow R_3, \quad \mu = -1$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2, R_4 \to R_4 - R_2 \quad \mu = 1$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$R_4 \to R_4 + R_3 \quad \mu = 1$

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -3 & 1 \end{pmatrix}$$

$R_3 \to R_3 + 2R_4, R_2 \to R_2 - R_4, R_1 \to R_1 - 2R_4 \quad \mu = 1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -2 & 6 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 4 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 & -3 & 1 \end{pmatrix}$$

So

$$A^{-1} = \begin{pmatrix} 1 & -2 & 6 & -2 \\ 0 & -1 & 4 & -1 \\ 0 & -1 & 4 & -2 \\ 0 & 1 & -3 & 1 \end{pmatrix}$$

and $(-1)det(A) = det(I) = 1$. So $det(A) = -1$.