Math 34600L Test 3 - Solutions Fall, 2025

1. (18 points) For the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(a) Compute a basis for the column space, $Col(A)$, and decide (with explanation) whether the columns span $\mathbb{R}^4$.

(b) Use the Gram-Schmidt procedure to obtain an orthonormal basis for $Col(A)$.

Put A in Reduced Echelon Form.

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$R_3 \to R_3 - R_1$:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$R_3 \to R_3 - R_2, \ R_4 \to R_4 - R_2$

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$R_3 \leftrightarrow R_4, \quad R_4 \to R_4 + 2R_3$:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_1 \to R_1 - R_3$:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So the basis is the first, second and fourth columns of A:

$$\{v_1 = (1,0,1,0)^T, v_2 = (0,1,1,1)^T, v_3 = (1,0,-1,1)^T\}.$$

Since $\mathbb{R}^4$ is has dimension 4, the columns do not span $\mathbb{R}^4$.
(b) For an orthogonal list, use $w_1 = v_1 = (1,0,1,0)^T$.

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = v_2 - \frac{1}{2} w_1 = (-\frac{1}{2}, 1, \frac{1}{2}, 1)^T.$$

or multiply by 2 to use $w_2 = (-1,2,1,2)$.

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = v_3 = (1,0,-1,1)^T.$$

To obtain an orthonormal basis, divide by the lengths to get

$$\{w_1/\sqrt{2}, w_2/\sqrt{10}, w_3/\sqrt{3}\}.$$

2. (16 points) Let $L : V \to W$ be a linear map between vector spaces.

(a) Define the Kernel of L and show that it is a subspace. Include a statement of which vector space it is a subspace.

(b) Assume that the zero vector 0 is the only vector in the kernel. Show that L is a one-to-one mapping, that is $L(v_1) = L(v_2)$ implies $v_1 = v_2$.

(a) The kernel of L is the subset of V consisting of those vectors which hit the zero vector in W.

$$Ker(L) \;=\; \{v \in V : L(v) = 0\}.$$

If v_1, v_2 are in $Ker(L)$, then $L(v_1) = L(v_2) = 0$ and so by linearity

$$L(v_1 + v_2) = L(v_1) + L(v_2) = 0 + 0 = 0.$$

Also for any scalar a, $L(av_1) = aL(v_1) = a0 = 0$.
Because the kernel is closed under vector addition and scalar multiplication it is subspace of V.

(b) If $L(v_1) = L(v_2)$, then $L(v_1 - v_2) = L(v_1) - L(v_2) = 0$ and so $v_1 - v_2$ is in the kernel. If 0 is the only vector in the kernel, then $v_1 - v_2 = 0$ and so $v_1 = v_2$.

3. (16 points) For the matrix

$$A \;=\; \begin{pmatrix} 3 & 1 & -2 \\ 0 & 2 & 2 \\ 0 & 2 & -1 \end{pmatrix}$$

(a) Compute the eigenvalues and a basis for each eigenspace.

(b) Compute an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. You need not compute Q^{-1}.

(a)

$$A - xI = \begin{pmatrix} 3-x & 1 & -2 \\ 0 & 2-x & 2 \\ 0 & 2 & -1-x \end{pmatrix}$$

So $det(A - xI) = (3-x)(x^2 - x - 2 - 4) = (3-x)(x-3)(x+2)$. The eigenvalues are $3, -2$.

With $\lambda = 3$

$$A - 3I = \begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{pmatrix} \quad \text{which is row equivalent to} \quad \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A basis for the null space is

$$\mathbf{e}_r = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{e}_s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

With $\lambda = -2$

$$A + 2I = \begin{pmatrix} 5 & 1 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad \text{which is row equivalent to} \quad \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

with eigenvector $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b)

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{with the diagonal matrix} \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

4. (18 points) Define the map $L : \mathcal{P}_2 \to \mathcal{P}_2$ (polynomials p of degree at most 2) by $L(p((x)) = p(5)x - 2p(x)$.

(a) Show that L is a linear map.

(b) For the bases $\mathcal{B} = \{1+x, 1-x, 1-x^2\}$ and $\mathcal{B}' = \{1, x, x^2\}$ for $\mathcal{P}_2$, compute the matrix $[L]_{\mathcal{B}'\ \mathcal{B}}$ of the linear map.

(c) The list $\mathcal{B}" = \{x, x-5, x^2-5x\}$ is also a basis for $\mathcal{P}_2$. Show that it is a basis of eigenvectors for L and compute the diagonal matrix $[L]_{\mathcal{B}"\ \mathcal{B}"}$.

(a) For polynomials $p_1(x), p_2(x)$ in $\mathcal{P}_2$,

$$L(p_1(x) + p_2(x)) = (p_1(5) + p_2(5))x - 2(p_1(x) + p_2(x)) = L(p_1(x)) + L(p_2(x)).$$

and $L(ap_1(x)) = ap_1(5)x - 2ap_1(x) = aL(p_1(x))$. Therefore L is a linear map.

(b)

$$[L]_{\mathcal{B}'\ \mathcal{B}} = ([L(1+x)]_{\mathcal{B}'}\ [L(1-x)]_{\mathcal{B}'}\ L(1-x^2)]_{\mathcal{B}'}] \ =$$

$$([6x - 2(1+x)]_{\mathcal{B}'}[-4x - 2(1-x)]_{\mathcal{B}'}[-24x - 2(1-x^2)]_{\mathcal{B}'}) = \begin{pmatrix} -2 & -2 & -2 \\ 4 & -2 & -24 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) $L(x) = 3x,\ L(x-5) = -2(x-5),\ L(x^2 - 5x) = -2(x^2 - 5x)$. So these are eigenvectors and the diagonal matrix of eigenvalues is $\begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

5. (16 points) Let $S = \{v_1, \dots, v_k\}$ be an orthogonal list of nonzero vectors in a Euclidean vector space with inner product $v_1 \cdot v_2$. Define the linear map $P_S : V \to V$ by:

$$P_S(v) \ = \ \sum_{i=1}^{k} \frac{v \cdot v_i}{v_i \cdot v_i} v_i.$$

Prove each of the following statements:

(a) The list S is li (linearly independent).

(b) For each vector v_j in the list S, $P_S(v_j) = v_j$.

(c) For each vector v in the span of S, $Span(S)$, $P_S(v) = v$.

(d) If $P_S(v) = 0$, i.e. v is in the kernel of P_S, then $v \cdot v_i = 0$ for every v_i in S.

(a) If $c_1v_1 + \dots c_kv_k = 0$, then we can take the inner product with v_j. Orthogonality means that $v_j \cdot v_i = 0$ if $i \neq j$ and so we are left with $c_j(v_j \cdot v_j) = v_j \cdot 0 = 0$. Since v_j is a nonzero vector, $v_j \cdot v_j > 0$ and so $c_j = 0$. Since this is true for all $j = 1, \dots, k$ all the coefficients are zero and so the list is li.

(b) Since $v_j \cdot v_i = 0$ if $i \neq j$, the coefficients of v_i equal zero in $P_S(v_j)$ except when $i = j$. So

$$P_S(v_j) \ = \frac{v_j \cdot v_j}{v_j \cdot v_j} v_j = v_j.$$

(c) If v is in the span of S, then for some coefficients $v = x_1v_1 + \dots x_kv_k$. By linearity of P_S we can apply part (b) to get

$$P_S(v) = x_1P_S(v_1) + \cdots + x_kP_S(v_k) = x_1v_1 + \cdots + x_kv_k = v.$$

(d) If $P_S(v) = 0$, then

$$0 = \sum_{i=1}^{k} \frac{v \cdot v_i}{v_i \cdot v_i} v_i.$$

By linear independence (part (a)) the coefficients $\frac{v \cdot v_i}{v_i \cdot v_i}$ are all zero. Because the denominators are positive, the numerators equal zero.

6. (16 points) (a) Define what it means for a matrix A to be symmetric and explain why it must be a square matrix.

(b) Assume that X_1 and X_2 are eigenvectors of a symmetric matrix A with distinct eigenvalues λ_1 and λ_2 (that is, $\lambda_1 \neq \lambda_2$. Show that the inner product $X_1 \cdot X_2 = 0$. (Recall that for column vectors the inner product is given by the matrix product $X_1^T X_2$.)

(c) Define what it means for a matrix P to be an orthogonal matrix.

(a) A matrix A is symmetric when it equals its transpose, that is, $A^T = A$. If A is an $m \times n$ matrix then A^T is $n \times m$ and so they can only be equal if $m = n$ and so the matrix is square.
(b) $X_1^T A X_2 = \lambda_2 X_1^T X_2$.

Since A is symmetric, we also have

$$X_1^T A X_2 = X_1^T A^T X_2 = (AX_1)^T X_2 = \lambda_1 X_1^T X_2.$$

So $(\lambda_1 - \lambda_2) X_1^T X_2 = 0$ and $(\lambda_1 - \lambda_2) \neq 0$, it follows that $X_1^T X_2 = 0$.

(c) A matrix P is orthogonal when it is invertible and $P^{-1} = P^T$, or, equivalently, $P^T P = I$.