Math 34600K Test 3 - Solutions Spring, 2024

1. (14 points) For the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 & 0 & -5 \\ 1 & 0 & -1 & 1 & 2 \\ 1 & 0 & -1 & 0 & -2 \\ 5 & 1 & -2 & 1 & -5 \end{pmatrix}$$

Compute a basis for and indicate the dimension of (a) the nullspace (the solution space of the homogeneous system), (b) the column space, and (c) the row space.

Put A in Reduced Echelon Form.

$R_1 \leftrightarrow R_3$:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 1 & 0 & -1 & 1 & 2 \\ 3 & 1 & 0 & 0 & -5 \\ 5 & 1 & -2 & 1 & -5 \end{pmatrix}$$

$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 3R_1, \quad R_4 \to R_4 - 5R_1$:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 1 & 3 & 1 & 5 \end{pmatrix}$$

$R_2 \leftrightarrow R_3$:

$$\begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 3 & 1 & 5 \end{pmatrix}$$

$R_4 \to R_4 - R_2 - R_3$:

$$Q = \begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

General solution of the homogeneous system is: $x_5 = r, x_4 = -4r, x_3 = s, x_2 = -3s - r, x_1 = s + 2r$. So a basis for the nullspace is

$$\mathbf{e}_r = \begin{pmatrix} 2 \\ -1 \\ 0 \\ -4 \\ 1 \end{pmatrix}, \quad \mathbf{e}_s = \begin{pmatrix} 1 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The basis for the column space consists of the first, second and fourth columns of A. The basis for the row space consists of the nonzero rows of Q.

2. (14 points) Let $L : V \to W$ be a linear map between vector spaces

(a) Define the Kernel of L and show that it is a subspace. Include a statement of which vector space it is a subspace.

(b) Assume that the zero vector θ is the only vector in the kernel. Show that L is a one-to-one mapping, that is $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ only when $\mathbf{v}_1 = \mathbf{v}_2$.

(a) The Kernel of L is the subset of V equal to $\{\mathbf{v} : L(\mathbf{v}) = \theta\}$.

If $L(\mathbf{v}) = L(\mathbf{w}) = \theta$, then $L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w}) = \theta$ and $L(c\mathbf{v}) = cL(\mathbf{v}) = \theta$. So the kernel is a subspace of V.

(b) If $L(\mathbf{v}_1) = L(\mathbf{v}_2)$, then $L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \theta$ and so $\mathbf{v}_1 - \mathbf{v}_2$ is in the kernel. If θ is the only vector in the kernel, then $\mathbf{v}_1 - \mathbf{v}_2 = \theta$ and so $\mathbf{v}_1 = \mathbf{v}_2$.

3. For the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}$$

(a) Compute the eigenvalues and a basis for each eigenspace.

(b) Compute an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. You need not compute Q^{-1}.

(c) Compute the general solution of the differential equation $\frac{dX}{dt} = AX$.

(a)

$$xI - A = \begin{pmatrix} x-2 & 1 & -2 \\ 0 & x-3 & 2 \\ 0 & -2 & x+2 \end{pmatrix}$$

So $det(xI - A) = (x-2)(x^2 - x - 6 + 4) = (x-2)(x-2)(x+1)$. The eigenvalues are $2, -1$.

With $l = 2$

$$2I - A = \begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \end{pmatrix} \quad \text{which is row equivalent to} \quad \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A basis for the null space is

$$\mathbf{e}_r = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{e}_s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

With $l = -1$

$$2I - A = \begin{pmatrix} -3 & 1 & -2 \\ 0 & -4 & 2 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{which is row equivalent to} \quad \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

with eigenvector $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

(b)

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{with the diagonal matrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(c)

$$X(t) = c_1 e^{2t} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

4. Define the linear map $L : \mathcal{P}_1 \to \mathcal{P}_2$ (polynomials p of degree at most 1 and at most 2,respectively) by $L(p)(x) = (x^2+1)p(1) + p(x)$. With respect to the basis $\mathcal{B} = \{x-1, 1\}$ for $\mathcal{P}_1$ and $\mathcal{B}' = \{x^2, x, 1\}$ for $\mathcal{P}_2$, compute the matrix $[L]_{\mathcal{B}'\ \mathcal{B}}$ of the linear map.

$L(x-1) = x-1, L(1) = x^2+2$. So

$$[L]_{\mathcal{B}'\ \mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 2 \end{pmatrix}.$$

5. The columns of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

are linearly independent.

Use the Gram-Schmidt procedure to obtain an orthonormal basis for the column space and use it to compute the orthogonal projection of the vector $\begin{pmatrix} 0 \\ 0 \\ -4 \\ 4 \end{pmatrix}$ onto the column space.

With the columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, the orthogonal list is given by

$$\mathbf{w}_1 = \mathbf{v}_1, \mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1,$$
$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2$$

$$\mathbf{w}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} - 0,$$

$$\mathbf{w}_3 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - 0 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}.$$

For the orthonormal basis we divide by the lengths so that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}\mathbf{w}_1, \quad \mathbf{u}_2 = \frac{1}{2}\mathbf{w}_2, \quad \mathbf{u}_3 = \frac{1}{\sqrt{2}}\mathbf{w}_3$$

$$P_U(\mathbf{v}) = (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v}\cdot\mathbf{u}_3)\mathbf{u}_3$$

With $\mathbf{v} = \begin{pmatrix}0\\0\\-4\\4\end{pmatrix}$,

$$P_U(\mathbf{v}) = 0 - (-4)\frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} - 0 = \begin{pmatrix}-2\\2\\-2\\2\end{pmatrix}$$

6. (14 points) Let $\{X_1, \ldots, X_k\}$ be a list of nonzero vectors in $\mathbb{R}^n$.
(a) Define what it means for the list to be orthogonal.
(b) Show that if the list is orthogonal, then it is linearly independent.

(a) The dot products $X_i \cdot X_j = (X_i)^T X_j$ all equal 0 when $i \neq j$.

(b) If $c_1X_1 + \ldots c_kX_k = \theta$ we must show all the c_i]s equal 0. The dot product with X_i is $c_i(X_i)^T X_i = 0$ and since X_i is not the zero vector, $(X_i)^T X_i > 0$ and so $c_i = 0$.

7. For the matrix

$$A = \begin{pmatrix}3 & 0 & 1 & 0\\0 & 3 & 0 & -1\\1 & 0 & 3 & 0\\0 & -1 & 0 & 3\end{pmatrix}$$

the eigenvalues are 4 and 2. Compute an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$. Compute the inverse of P as well (Hint: P is orthogonal).

With $l = 4$

$$4I - A \quad = \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{is row equivalent to} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis for the null space is

$$\mathbf{e}_r = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{e}_s = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

With $l = 2$

$$2I - A \quad = \quad \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad \text{is row equivalent to} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis for the null space is

$$\mathbf{e}_r = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{e}_s = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The four vectors form an orthogonal list. To obtain P we need an orthonormal list and so we divide by the lengths which are all $\sqrt{2}$. So

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and the diagonal matrix is} \quad \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Because P is orthogonal,

$$P^{-1} = P^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$