1. (16 points) We use the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$.

(a) Show that $B = \begin{pmatrix} 1 & 0 & -1 & 1 \end{pmatrix}^T$ is a linear combination of the columns of A and find a basis for the column space of A

(b) Find a basis for the row space of A.

(a) To obtain B are a linear combination of the columns of A, we put the augmented matrix $(A|B)$ is Reduced Row Echelon Form.

$$(A|B) = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & | & 1 \\ 2 & 6 & 4 & 6 & 6 & | & 0 \\ 0 & 0 & 0 & 0 & 1 & | & -1 \\ 0 & 0 & 2 & 2 & 1 & | & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 4 & 4 & 6| & -2 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to \frac{1}{2}R_2$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 2 & 2 & 3| & -1 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to R_2 - R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 2| & -2 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 1| & 1 \end{pmatrix}$$

$R_2 \to R_2 - 2R_3, R_4 \to R_4 - R_3$.

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 0| & 0 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 2 & 2 & 0| & 2 \end{pmatrix}$$

$R_4 \to \frac{1}{2}R_4$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 0| & 0 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 1 & 1 & 0| & 1 \end{pmatrix}$$

$R_2 \leftrightarrow R_4$

$$(Q|\tilde{B}) = \begin{pmatrix} 1 & 3 & 0 & 1 & 0| & 1 \\ 0 & 0 & 1 & 1 & 0| & 1 \\ 0 & 0 & 0 & 0 & 1| & -1 \\ 0 & 0 & 0 & 0 & 0| & 0 \end{pmatrix}$$

The system is consistent and the general solution is given by $x_5 = -1, x_4 = r, x_3 = 1-r, x_2 = s, x_1 = 1-3s-r$. So B is a linear combination of the columns of A.

For A put in reduced echelon form, obtaining Q, there are leading ones in the first, third and fifth columns. So the basis is given by the first, third and fifth columns of the original matrix A

(b) The three nonzero rows of Q provide a basis for the row space.

2. (24 points) Assume that $\{v_1, \ldots, v_n\}$ is a list of vectors in a vector space V.

(a) Define what it means that the list $\{v_1, \ldots, v_n\}$ is linearly independent.

(b) Show that if v_n is a linear combination of $\{v_1, \ldots, v_{n-1}\}$, then the list $\{v_1, \ldots, v_n\}$ is linearly dependent, i.e. not linearly independent.

(c) Show that if $\{v_1, \ldots, v_{n-1}\}$ spans V, then $\{v_1, \ldots, v_{n-1}, v_n\}$ is linearly dependent.

(d) Show that a list of two vectors $\{v_1, v_2\}$ is linearly dependent if and only if one of the vectors is a multiple of the other.

(a) $\{v_1, \ldots, v_n\}$ is li when $c_1v_1 + \cdots + c_nv_n = \theta$ only when $c_1 = \cdots = c_n = 0$.

(b) If $v_n = x_1v_1 + \ldots x_{n-1}v_{n-1}$, then $\theta = x_1v_1 + \ldots x_{n-1}v_{n-1} + (-1)v_n$. The coefficient of v_n is not zero.

(c) If $\{v_1, \ldots, v_{n-1}\}$ spans V, then v_n is a linear combination of $\{v_1, \ldots, v_{n-1}\}$, and so the list $\{v_1, \ldots, v_n\}$ is linearly dependent as in (b).

(d) If $v_2 = Cv_1$, then as in (b) $\theta = Cv_1 + (-1)v_2$. If $\theta = c_1v_1 + c_2v_2$ and $c_2 \neq 0$, then $v_2 = Cv_1$ with $C = -c_1/c_2$.

3. (16 points) For an $m \times n$ matrix A, the null space, $Null(A) = \{X \in \mathbb{R}^n : AX = \theta\}$ where θ is the 0 matrix. That is, $Null(A)$ is the solution space of the associated homogeneous system.

(a) Show that $Null(A)$ is a subspace of R^n.

(b) For the matrix $A = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 4 & 6 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \end{pmatrix}$ given in problem 1, find a basis for $Null(A)$. (You may use your work from problem 1.)

(a) If $AX = \theta$ and $AY = \theta$ then $A(X+Y) = AX + AY = \theta + \theta = \theta$ and $A(cX) = c(AX) = c\theta = \theta$. So the nullspace is a subspace.

(b) Using our work from (1) We put A is Reduced Row Echelon Form to obtain $Q = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Solution of the homogeneous system is : $x_5 = 0, x_4 = r, x_3 = -r, x_2 = s, x_1 = -3s - r$. The basis vectors are $e_r = (-1, 0, -1, 1, 0), e_s = (-3, 1, 0, 0, 0)$.

4. (16 points) Assume that A is an $m \times n$ matrix.

(a) Assume that the columns of A form a linearly independent list. What is the rank of A? (Explain)

If, in addition, $m = n$ so that the matrix is square, does this imply that A has an inverse? (Explain)

(b) If $m < n$, can the columns form a linearly independent list? Can the columns span $\mathbb{R}^m$? (Explain each answer)

(a) If Q is in Reduced Row Echelon Form row equivalent to A then the columns are linearly independent exactly when every column of Q has a leading 1 and so when the rank equals n.

If $m = n$, then $Q = I$ and A is invertible.

(b) If $m < n$ then the largest the rank can be is m which is less than n and so the columns cannot be linearly independent. However, if the rank is m, then there is a leading 1 in every row of Q and so the columns do span in that case.

5. (16 points) Consider the following list of polynomials in $\mathcal{P}_3$, the vector space of polynomials of degree at most three.

$$S = \{t^3 + 2t^2, 3t^3 + 6t^2, 4t^2 + 2, t^3 + 6t^2 + 2, 6t^2 + t + 1\}.$$

(a) Show that the list S is linearly dependent.

(b) Obtain a basis for the $Span(S)$ and compute the dimension of $Span(S)$.

(a) Observe that $3t^3 + 6t^2 = 3(t^3 + 2t^2$ so the list is ld. Alternatively, proceed to (b)

(b) Translating the polynomials into column vectors we obtain the matrix A given in problems 1 and 3. In reduced row echelon form we obtain as before

$$Q = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is not a leading one in every column, the list of columns is linearly dependent. Also we saw that a basis for the column space consisted of the first, third and fifth columns of A. Translating back to polynomials, a basis for $Span(S)$ is

$$\{t^3 + 2t^2, 4t^2 + 2, 6t^2 + t + 1\}.$$

6. (12 points)(a) For a vector space V, define "the dimension of V".
(b) Using the definition of dimension, show that the dimension of $\mathbb{R}^3$ is 3.

(a) The dimension of V is the number of vectors in any basis.

(b)The columns of $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ provide a basis for $\mathbb{R}^3$ consisting of three vectors and so the dimension is 3.