MATH 346 Test 1 Solutions Spring, 2024

1. (15 points) If $A = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 3 & 0 \\ 1 & 1 \end{pmatrix}$, compute each of the following if it can be done. If it cannot be done, If it cannot be done, explain why not.

$$(i)\ \ AB \qquad (ii)\ \ AB^T \qquad (iii)\ \ A+B \qquad (iv)\ \ A-A^T.$$

(i) can't multiply a 2×2 by a 3×2.

(ii) $AB^T = \begin{pmatrix} 14 & 9 & 2 \\ 11 & 6 & 3 \end{pmatrix}$.

(iii) can't add with different shapes.

(iv)$A - A^T = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$

2. (15 points) (a) For the row operation: To Row 2 add a copy of 5 times Row 3, write down the associated 3×3 elementary matrix E and its inverse E^{-1}.

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Assume that Q is a 3×3 matrix in reduced echelon form. If the rank of Q is 3, what is true of Q? If the rank of Q is not 3, what is true of Q? [Fredholm Alternative]

If the rank equals 3 then $Q = I_3$.

If the rank does not equal 3 then Q has a row of zeroes at the bottom.

3. (18 points)(a) For A a 7×5 matrix in echelon form, what is the largest rank A can have? What can you say about the bottom row? (Explain)

The rank is at most 5 since every column has at most one leading 1. So there are at least two rows without leading 1's and these bottom two rows are rows of zeroes.

(b) When can a 3×3 matrix B have rank zero? (Explain)

If a matrix in echelon form has no leading 1's then every row is a zero row and so the matrix equal 0_3. Since B is row equivalent to the zero matrix it must be the zero matrix.

(c) If C a 5×5 matrix with $det(C) = 3$, what is $det(2C)$?

$det(2C) = 2^5 det(C) = 32 \cdot 3 = 96$.

4. (18 points)(a) If A is an $n \times n$ matrix define what it means for A to be *invertible*.

A is invertible when there exists an $n \times n$ matrix B which cancels A. That is, $AB = I_n = BA$.

(b) If A is an $n \times n$ matrix with a row of zeroes, show that A is not invertible.

If row i of A is a row of zeroes then for any $n \times n$ matrix B, row i of AB is a row of zeroes and so $AB \neq I_n$.

(c) If A and B are invertible $n \times n$ matrices, show that the product AB is invertible and compute its inverse.

$(AB)(B^{-1}A^{-1}) = I_n$ and $(B^{-1}A^{-1})(AB) = I_n$. Since $(B^{-1}A^{-1})$ cancels AB it is the inverse of AB.

5. (18 points)(a) Put the matrix

$$A \quad = \quad \begin{pmatrix} 3 & 1 & 0 & 0 & -5 \\ 1 & 0 & -1 & 1 & 2 \\ 1 & 0 & -1 & 0 & -2 \\ 2 & 1 & 1 & -1 & -7 \end{pmatrix}$$

in *reduced echelon form* and compute its *rank*.

$$R_1 \Leftrightarrow R_2 : \begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 3 & 1 & 0 & 0 & -5 \\ 1 & 0 & -1 & 0 & -2 \\ 2 & 1 & 1 & -1 & -7 \end{pmatrix}$$

$$R_2 - 3R1, R_3 - R_1, R_4 - 2R_1 : \begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 3 & -3 & -11 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 1 & 3 & -3 & -11 \end{pmatrix}$$

$$R_4 - R_2, (-1)R_3 : \begin{pmatrix} 1 & 0 & -1 & 1 & 2 \\ 0 & 1 & 3 & -3 & -11 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_2 + 3R_3, R_1 - R_3 : \begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank $= 3$.

(b) Compute the general solution for the system of equations which has A as its augmented matrix.

$x_4 = 4, \quad x_3 = r, \quad x_2 = 1 - 3r, \quad x_1 = -2 + r.$

6.(16 points) Compute the inverse and the determinant of each of the following matrices:

(a)

$$A \quad = \quad \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$$

$det(A) = 6 - -5 = 11. \qquad A^{-1} = \frac{1}{11}\begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}.$

(b)

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \leftrightarrow R_2, \quad \mu = -1$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$R_2 - 2R_1, R_3 - R_1, R_4 - 3R_1, \quad \mu = 1$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 4 & 0 & -3 & 0 & 1 \end{array}\right)$$

$R_4 - R_2, \quad \mu = 1$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 0 & 1 \end{array}\right)$$

$R_3 \leftrightarrow R_4 \quad \mu = -1$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{array}\right)$$

$R_3 - 2R_4, R_2 - 2R_4, R_1 + R_4, \quad \mu = 1$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{array}\right)$$

$R_2 + 2R_3, R_1 - R_3, \quad \mu = 1$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 3 & -1 \\ 0 & 1 & 0 & 0 & -1 & 2 & -6 & 2 \\ 0 & 0 & 1 & 0 & -1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \end{array}\right)$$

$A^{-1} = \begin{pmatrix} 1 & -1 & 3 & -1 \\ -1 & 2 & -6 & 2 \\ -1 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$ $det(A)$ is the reciprocal of the product of the multipliers μ and so equals 1.