Math 346 Study Guide - Spring, 2024

Part I

(1) Matrix Arithmetic - Be able to perform matrix computations. Know when the sum or product of two matrices can be defined. Know the two interpretations of $AX = B$ for an $m \times n$ matrix A. (m equations in n unknowns; linear combination of the columns of A)

(2) Matrix Inverse and Transpose - know the definition and properties.
(a) If a matrix A has a row of zeroes, can it have an inverse? Explain.
(b) If A and B are invertible $m \times m$ matrices, prove that AB is invertible.
(c) When is a matrix symmetric? For any matrix A show that AA^T is defined and is symmetric.

(3) Echelon and Reduced Echelon Form - Know the meaning and be able to perform the steps to put a matrix in echelon form.

Be able to compute the rank of a matrix.

(4) Elementary Matrices - Know the definition of the elementary matrices associated with a row operation.
Know that doing a row operation is the same as multiplying by the corresponding elementary matrix.
Explain why an elementary matrix is invertible.
If A is an $m \times m$ matrix row equivalent to Q in reduced echelon explain what is true about Q when the rank equals m or when the rank is less than m. In any case, explain why $Q = UA$ with U an invertible matrix.

(5) Solve the system $AX = B$ by using the augmented matrix $(A|B)$.
Explain why a system can have either 0 or 1 or infinitely many solutions. When do the different possibilities occur?

(6) Compute the inverse of A by using $(A|I)$ and compute the inverse directly if A is a 2×2 matrix.
Using the inverse of A, when it exists, solve the system $AX = B$.

(7) Linear Transformation - When is a function $T : \mathbb{R}^n \to R^m$ a linear transformation?
Explain why $T_A(X) = AX$ is such a linear transformation.
Using the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ for $\mathbb{R}^m$ show that any linear transformation $T : \mathbb{R}^n \to R^m$ comes from a matrix A in this way.

(8) Determinants - know the properties and computation using reduction to echelon form.

Part II

(1) Subspace of a Vector Space - know the definition and compute examples.

(2) For a list $\{v_1, v_2, \ldots, v_n\}$ in a vector space V know the definition and important properties of the following:

(a) linear combination.

(b) linear independence and linear dependence.

(c) the span of a list.

(d) when the list spans, i.e. spans the entire space, or when a list spans a subspace.

(e) basis of a space or subspace.
(f) dimension of a space or subspace.
(g) coordinates of a vector with respect to a basis.

(3) For an $m \times n$ matrix A be able to define and compute a basis for:

(a) The Null Space of A (the solution space of the homogeneous system).
(b) The Column Space.
(c) The Row Space.

(4) For an $m \times n$ matrix A row equivalent to Q in Reduced Echelon Form, understand the following equivalences:

(a)The columns of A form a linearly independent list.
(b) Every column of Q has a leading 1.
(c) The rank of A equals n.

and understand the following equivalences:

(a) The columns of A span $\mathbb{R}^m$.
(b) Every row of Q has a leading 1.
(c) The rank of A equals m.

and when $m = n$ so that A is a square matrix understand the following equivalences:

(a) A is invertible.
(b) $Q = I$.
(c) The rank of A equals n.
(d) The columns form a basis for $\mathbb{R}^n$.

Part III

(1) Know what it means for a map $T : V \to W$ between vector spaces to be linear.
Know the definition of the kernel and the image of T and be able to show they are subspaces.
Know: T is one-to-one if and only if $Ker(T) = \{0\}$.
T is onto if and only if $Im(T) = W$.
If T is linear, one-to-one and onto, then the inverse map $T^{-1} : W \to V$ is linear. This is called an isomorphism.
For an $m \times n$ matrix A, define the linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and identify the kernel and image of T_A.

(2) For a list a list $D = \{v_1, v_2, \ldots, v_n\}$ in a vector space V, define $T_D : \mathbb{R}^n \to V$ by $T_D(x_1, \ldots, x_n) = x_1 v_1 + \cdots + x_n v_n$. Be able to show:

(a) T_D is a linear map.

(b) $Ker(T_D) = \{0\}$ if and only if the list D is linearly independent,

(c) $Im(T_D) = V$ if and only if the list D spans V.

(d) T_D is an isomorphism if and only if the list D is a basis for V.

(3) Given a linear map $T : V \to W$, a basis B for V and a basis D for W know the definition of the associated matrix $[T]_{DB}$. Be able to compute it (especially when D is a standard basis).

(4) Know the definition of an eigenvector for an $n \times n$ matrix and its associated eigenvalue. Be able to show that if X_1, X_2, X_3 are eigenvectors of A with distinct eigenvalues (no two equal), then the list $\{X_1, X_2, X_3\}$ is linearly independent.

(5) Given a square matrix A be able to compute the eigenvalues as the roots of the characteristic equation and be able to obtain a

basis for each associated eigenspace. If there are sufficiently many to obtain a basis of the space consisting of eigenvectors, use them to construct a matrix P such that $P^{-1}AP$ is a diagonal matrix and identify the diagonal matrix.

(6) Be able to obtain the general solution of the system $\frac{dX}{dt} = AX$ when A is diagonalizable.

(7) Be able to describe when a list $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is an orthogonal list or an orthonormal list. Be able to show that an orthogonal list of nonzero vectors is linearly independent.

(4) Given a basis for a subspace U of $\mathbb{R}^n$ be able to use the Gram-Schmidt process to obtain an orthonormal basis for U and use it to compute the orthogonal projection of a vector X on U.

(5) Be able to define when a square matrix is orthogonal and when it is symmetric. Be able to show that if A is a symmetric matrix and X_1, X_2 are eigenvectors with eigenvalues $\lambda_1 \neq \lambda_2$, then X_1 and X_2 are perpendicular. That is, $X_1^T X_2 = 0$.

(6) Given a symmetric matrix be able to obtain an orthonormal basis of eigenvectors and use them to construct an orthogonal matrix P such that such that $P^T AP = P^{-1}AP$ is a diagonal matrix and identify the diagonal matrix.