SUPPLEMENTARY NOTES FOR MATH 212

Chapter and section numbers refer to both 14th and 15th edition of the textbook:

Thomas' Calculus: Early Transcendentals, Haas, Heil, and Weir (Pearson)

Review of Chapter 5

From first semester calculus, students are expected to know three techniques (theorems) for evaluating integrals.

1 Linearity. Integration is linear, viz., for a constant c and continuous functions $f(x)$ and $g(x)$,

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

2 Substitution Theorem. For continuous functions $f(x)$, $u(x)$ and $g(x)$, $\int f(u(x))u'(x)\,dx = \int f(u)\,du$. The meaning of the right side of this equality is that if F is an antiderivative of f, then $\int f(u)\,du = F(u(x))$.

3 The Fundamental Theorem of Calculus. If $f(x)$ is a differentiable function and a is a constant, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{and} \quad \int_a^x \frac{df}{dt}\,dt = f(x) - f(a).$$

(The Fundamental Theorem could be stated more generally, but this is is sufficient to produce entries in a table of integrals in Section 8.1)

Example: Since $\frac{d}{dx}\sin x = \cos x$, we have $\int \cos x\,dx = \sin x + C$.

Example: Evaluate $\int x(x^2-4)^9\,dx$.

Solution 1: We rewrite the integral in a form for which the substitution theorem applies, with $f(x)=x^9$, and $u(x)=x^2-4$.

$$\int x(x^2-4)^9\,dx = \int \frac{1}{2}2x(x^2-4)^9\,dx = \frac{1}{2}\int 2x(x^2-4)^9\,dx$$
$$=\frac{1}{2}\int u^9\,du = \frac{1}{2}\frac{u^{10}}{10} = \frac{(x^2-4)^{10}}{20}+C$$

Solution 2 (which shows why the theorem is named the Substitution Theorem): The symbol $\frac{df}{dx}$ means the derivative of f but we treat it as if it were a fraction.

$$\begin{aligned}&\int x(x^2-4)^9\,dx \\ &= \int u^9(\frac{1}{2}\,du) \\ &= \frac{1}{2}\frac{u^{10}}{10} \\ &= \frac{(x^2-4)^{10}}{20}+C\end{aligned}$$

$$\begin{aligned} u &= x^2-4 \\ \frac{du}{dx} &= 2x \\ du &= 2x\,dx \\ \frac{1}{2}du &= x\,dx \end{aligned}$$

For a definite integral, one must distinguish between limits for x and limits for u. We illustrate with the function above integrated from 2 to 3. As x ranges from 2 to 3, $u(x)$ ranges from $u(2)=0$ to $u(3)=5$, so

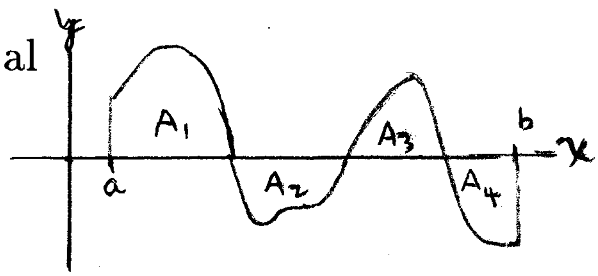
$$\int_{x=2}^{3} x(x^2-4)^9\,dx = \int_{u=0}^{5} u^9\,du = \frac{u^{10}}{20}\Big|_0^5 = \frac{5^{10}}{20} \quad \text{or}$$
$$\int_{x=2}^{3} x(x^2-4)^9\,dx = \frac{(x^2-4)^{10}}{20}\Big|_2^3 = \frac{5^{10}}{20}.$$

but **NOT** $\frac{u^{10}}{\ } \Big|_2^3 = \frac{3^{10}-2^{10}}{\ }$

Some elementary facts about integration are listed below.

1. For constants a, b and c, $\int_a^b c\,dx = c(b-a)$.

2. $\int_a^b f(x)\,dx$ is equal to the total area below the curve when it is above the x-axis minus the total area above the curve where it is below the x-axis. For example, for the function whose graph is on the right, the value of the integral is $A_1 + A_3 - (A_2 + A_4)$.

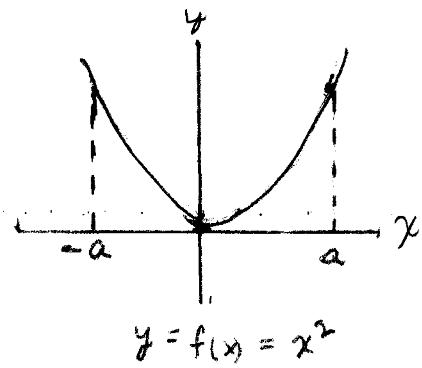


3. For $a \leq b$ and $\int_a^b f(x)\,dx = F(x)|_a^b$,
$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = F(x)|_b^a$.

Example: Instead of
$\int_a^b \sin x\,dx = -\cos x|_a^b = -\cos b - (-\cos a) = \cos a - \cos b$
one can write
$\int_a^b \sin x\,dx = -\cos x|_a^b = \cos x|_b^a = \cos a - \cos b$.

4. A function f is called *even* if $f(-x) = f(x)$ for all x in it's domain; $f(x) = x^2$ is even. A function is even if and only if it has symmetry about the y-axis.

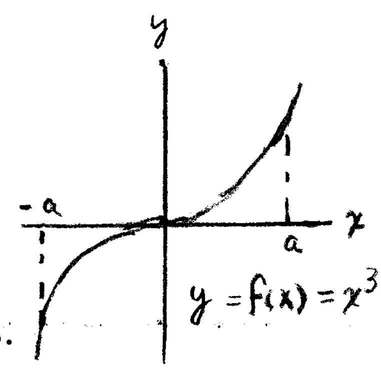
If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.



A function f is called *odd* if $f(-x) = -f(x)$ for all x in it's domain; $f(x) = x^3$ is odd. A function is odd if and only if it has symmetry about the origin.

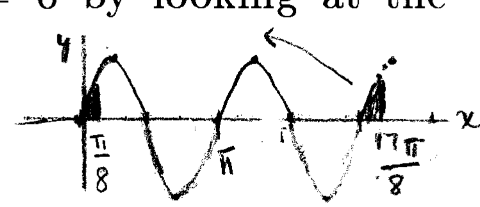
If f is odd, $\int_{-a}^{a} f(x)\,dx = 0$.

A product of two even or two odd functions is even; a product of an even and an odd function is odd. A polynomial is even (odd) if and only if only even (odd) power terms have nonzero coefficients.



5. $\int_a^b \sin(cx)\,dx = 0$ and $\int_a^b \cos(cx)\,dx = 0$ if the length $b - a$ of the interval of integration is an integral multiple of $2\pi/c$. We explain why. First, the period $\sin(cx)$ and $\cos(cx)$ is found by setting $cx = 2\pi$, so the period is $2\pi/c$. Now we illustrate with an example.

Example: Show $\int_{\pi/8}^{17\pi/8} \sin 2x\,dx = 0$ by looking at the graph. The period of $\sin 2x$ is $2\pi/2 = \pi$. By looking at the graph, we see we could move the portion of the graph from 2π to $17\pi/8$ back to the origin to obain two loops of the graph above the x-axis and two loops below the x-axis, and this cut and paste from the end to the beginning works because the



Section 7.1

For integration (and differentiation) of exponential and logarithmic functions, the following change of base formulas can be useful:

$$\log_b c = \frac{\log_a c}{\log_a b} \qquad\qquad b^c = a^{c \log_a b}$$

Use of the special case $a = e$ occurs frequently:

$$\log_b c = \frac{\ln c}{\ln b} \qquad\qquad b^c = e^{c \ln b}$$

Section 7.2

Exponential Growth and Decay

Quantities y whose size can be described by the equation $y(t) = Ae^{kt}$, where A and k are constants are said grow (if $k > 0$) or decay (if $k < 0$) *exponentially*. Letting $t = 0$ in this equation shows $y(0) = A$. Often y_0 is used to denote $y(0)$.

Using the change of base formula $e^{kt} = b^{kt \log_b e}$, we see that the equation above can be described using any base. Frequently, using base e is convenient, but there are times when another base is convenient.

For two values, t_1 and t_2, of t, if we know the values $y_i = y(t_i)$, and we let $\Delta t = t_2 - t_1$, then

$$\frac{y_2}{y_1} = \frac{y_0 e^{kt_2}}{y_0 e^{kt_1}} = e^{k\Delta t}$$

$$\ln\left(\frac{y_2}{y_1}\right) = k\Delta t$$

$$k = \frac{1}{\Delta t} \ln\left(\frac{y_2}{y_1}\right)$$

$$y(t) = y_0 e^{\left[\frac{1}{\Delta t}\ln(y_1/y_2)\right]t} = y_0\left(e^{\ln(y_2/y_1)}\right)^{t/\Delta t}$$

$$y(t) = y_0\left(\frac{y_2}{y_1}\right)^{t/\Delta t},$$

and this form of expression of $y(t)$ is also sometimes convenient.

Example: Suppose a group of rabbits initially has 60 rabbits and the number of rabbits in the group doubles every 8 days.

(a) Find a formula for the approximate number, $N(t)$, of rabbits after t days.

(b)Find the approximate number of rabbits after 44 days.

(c) Find the number (the answer is an integer) of rabbits after 48 days.

Solution: (a) $y_0 = 60$, $y_8 = 2(60)$, so

$$y(t) = y_0 \left(\frac{y_8}{y_0}\right)^{t/8} = (60)2^{t/8}.$$

(b) $$\begin{aligned} y(44) =&(60)2^{44/8} = (60)2^{5+1/2} = (60)(32)\sqrt{2} \\ =&1920\sqrt{2} \approx 2715. \end{aligned}$$

(c) $y(48) = (60)2^6 = (60)(64) = 3840.$

Note: Applying the change of base formula in (a) we get $y(t) = 60e^{[(\ln 2)/8]t}$, and in part (c), we would get $y(48) = 60e^{6\ln 2}$, which does simplify to $(60)(64)$, but perhaps this is less obvious.

Example 3 from section 7.2 in text: A yeast culture originally has 29 g., and 30 minutes later has 37 g. How long does it take to double.

Solution: $y(t) = 29\left(\frac{37}{29}\right)^{t/30}$. Let t_2 be the time to double from 29 g to 58 g:

$$y(t_2) = 29\left(\frac{37}{29}\right)^{t_2/30} = 58$$
$$\left(\frac{37}{29}\right)^{t_2/30} = 2$$
$$(t_2/30)\ln\left(\frac{37}{29}\right) = \ln 2$$
$$t = \frac{30\ln 2}{\ln(37/29)}.$$

Example: All temperatures are in degrees Fahrenheit. An object is submerged in a liquid maintained at 60°. One hour after being submerged the temperature of the object is 180°, and after three hours, the temperature is 90°. Let $T(t)$ be the temperature of the object at time t hours, and let $(\Delta T)(t) = T(t) - 60$ be the difference between the temperature of the object and the temperature of the liquid. Assume that $(\Delta T)(t)$ satisfies an exponential decay law.

(a) Find the function $T(t)$.

(b) Find, to the nearest half degree without using a calculator or other electronic device, the temperature of the object 6 hours after being submerged.

(c) Find an exact expression for the time t at which the temperature of the object is 61°.

Solution: In tabular form the information given is

t	$T(t)$	$(\Delta T)(t)$
0	?	
1	180	120

So

$$\begin{aligned}
\text{(a)}\quad (\Delta T)(t) &= (\Delta T)_0 \left(\frac{30}{120}\right)^{t/(3-1)} \\
&= (\Delta T)_0 \left(\frac{1}{4}\right)^{t/(3-1)} = (\Delta T)_0 \left(\frac{1}{2}\right)^{t}. \\
120 = (\Delta T)(1) &= (\Delta T)_0 \left(\frac{1}{2}\right)^{1} \\
(\Delta T)_0 &= 240 \\
T(t) &= 60 + (\Delta T)(t) = 60 + 240\left(\frac{1}{2}\right)^{t}
\end{aligned}$$

$$\text{(b)}\quad T(6) = 60 + 240\left(\frac{1}{2}\right)^{6} = 60 + \frac{240}{64} = 60 + \frac{15}{4} = 63.75$$

$$\begin{aligned}
\text{(c)}\quad 61 = T(t) &= 60 + (240)\left(\frac{1}{2}\right)^{t} \\
1 &= 240\left(\frac{1}{2}\right)^{t} \\
\left(\frac{1}{2}\right)^{t} &= \frac{1}{240} \\
t \ln \frac{1}{2} &= \ln \frac{1}{240} \\
t &= \frac{\ln \frac{1}{240}}{\ln \frac{1}{2}} = \frac{\ln 240}{\ln 2}. \qquad (\approx 7.9 \text{ hours})
\end{aligned}$$

Section 8.8

Rate of Growth of Functions

The material presented here is similar to that in Section 7.4. It is useful when applying the Limit Comparison for improper integrals, and a discrete analogue of this material will be essential for much of what is done in Chapter 10 on Infinite Series.

By the phrase "$f = o(g)$ at a" we mean $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$. Usually, we will be interested in $a = \infty$ and functions f and g for which $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Our intuitive interpretation of $f = o(g)$ is that the function g is getting large faster than f as $x \to \infty$.

For now lim will mean $\lim_{x\to\infty}$.

Example 1: If $0 < m < n$, then $x^m = o(x^n)$, i.e.,

$$\lim \frac{x^m}{x^n} = \lim \frac{1}{x^{n-m}} = 0.$$

For a more specific example x^2 and x^3 both get large as x becomes large; 100^2 is ten thousand, quite large, but 100^3 is a million, much larger.

The notation used here, instead of $f = o(g)$, is $f << g$. This is done to call attention to the fact that the relation is transitive: if $f << g$ and $g << h$, then $f << h$. The proof is simply that

$$\lim \frac{f(x)}{h(x)} = \lim \frac{f(x)}{g(x)} \frac{g(x)}{h(x)} = 0.$$

If f and g are non-negative functions, then $\lim(f/g) = 0$ if and

Example 2: By L'Hôpital's rule, $\lim \frac{x}{e^x} = \lim \frac{1}{e^x} = 0.$ Now $\lim \frac{x^2}{e^x} = \lim \frac{2x}{e^x} = 0$. In the same fashion if $p(x)$ is any polynomial, then $p(x) << e^x$. If $0 < a \leq 1$ and $x \geq 1$, then $x^a \leq x$, so $x^a << e^x$. For $a > 1$, continued iteration of

$$\lim \frac{x^a}{e^x} = \lim \frac{ax^{a-1}}{e^x} = \ldots$$

leads to the conclusion that $x^a << e^x$, for all $a > 0$.

Example 3: For $a > 0$,

$$\lim \frac{\ln x}{x^a} = \lim \frac{1/x}{ax^{a-1}} = \lim \frac{1}{ax^a} = 0,$$

i.e. $\ln x << x^a$.

Reasoning as in Examples 1–3, we see that if:

log is $\log_b x$, $b > 1$;
$p(x)$ is a polynomial, or a positive power of x; and
exp is an exponential a^{cx}, $a > 1$, $c > 0$,

then $\log << p(x) << \exp$.

If U, V, u_i, v_i, for all i, are non-negative functions and

$f = U + u_1 + \cdots + u_m$, where $u_i << U$ for all i;
$g = V + v_1 + \cdots + v_n$, where $v_i << V$ for all i; and $U << V$,

then, by dividing both numerator and denominator by V, we see

$$\lim \frac{f}{g} = \lim \frac{\frac{U}{V} + \frac{u_1}{V} + \cdots \frac{u_m}{V}}{1 + \frac{v_1}{V} + \cdots \frac{v_n}{V}} = \lim \frac{U}{V}.$$

Example 4:

$$\lim \frac{x^4 + 3x^3 + 4}{x^6 + 6x^2 + 1} = \lim \frac{x^4}{x^6} = 0.$$

Example 5:

$$\lim \frac{e^x + x^2 + 1}{x^8 + \ln x} = \lim \frac{e^x}{x^8} = \infty.$$