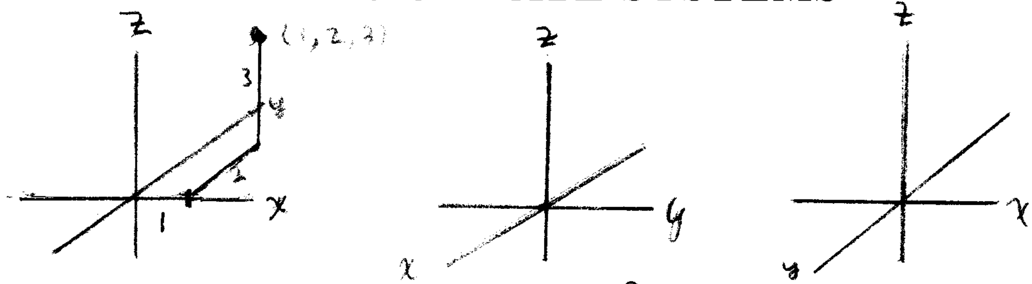
Section 12.1

3-Dimensional Coordinate Systems

3-DIMENSIONAL COORDINATE SYSTEMS



We will refer to the plane as $\mathbf{R}^2$ and all of 3-dimensional space as $\mathbf{R}^3$.

Points in 3-dimensional space are described in a manner analogous to points in the plane. We draw three mutually perpendicular lines, again called axes, each of which is considered to be a copy of the real line. In the diagram above, the axes, labelled $x-$, $y-$ and $z-$axes, have their letter names written on the positive side of the real line. The intersection of these axes is called the origin. The planes that go through two axes are called *coordinate planes*. For example the horizontal plane that goes through the $x-$ and $y-$axis is called the $xy-$plane. It is common practice to call the vertical axis the z-axis. The coordinates of a point in three dimensions, (a, b, c), identify a point by starting at the origin and travelling a amount in the x direction, then b amount in the y direction, and c amount in the z direction.

The left and middle diagrams above are the same, in the sense that either one could be rotated about the $z-$axis to look exactly like the other. On the other hand, the diagram above on the right can not be rotated to look like the other two. Every 3-dimensional coordinate system is either right-handed or left-handed, but not both. We describe this on a separate slide. WE WILL ALWAYS USE A RIGHT HANDED COORDINATE SYSTEM.

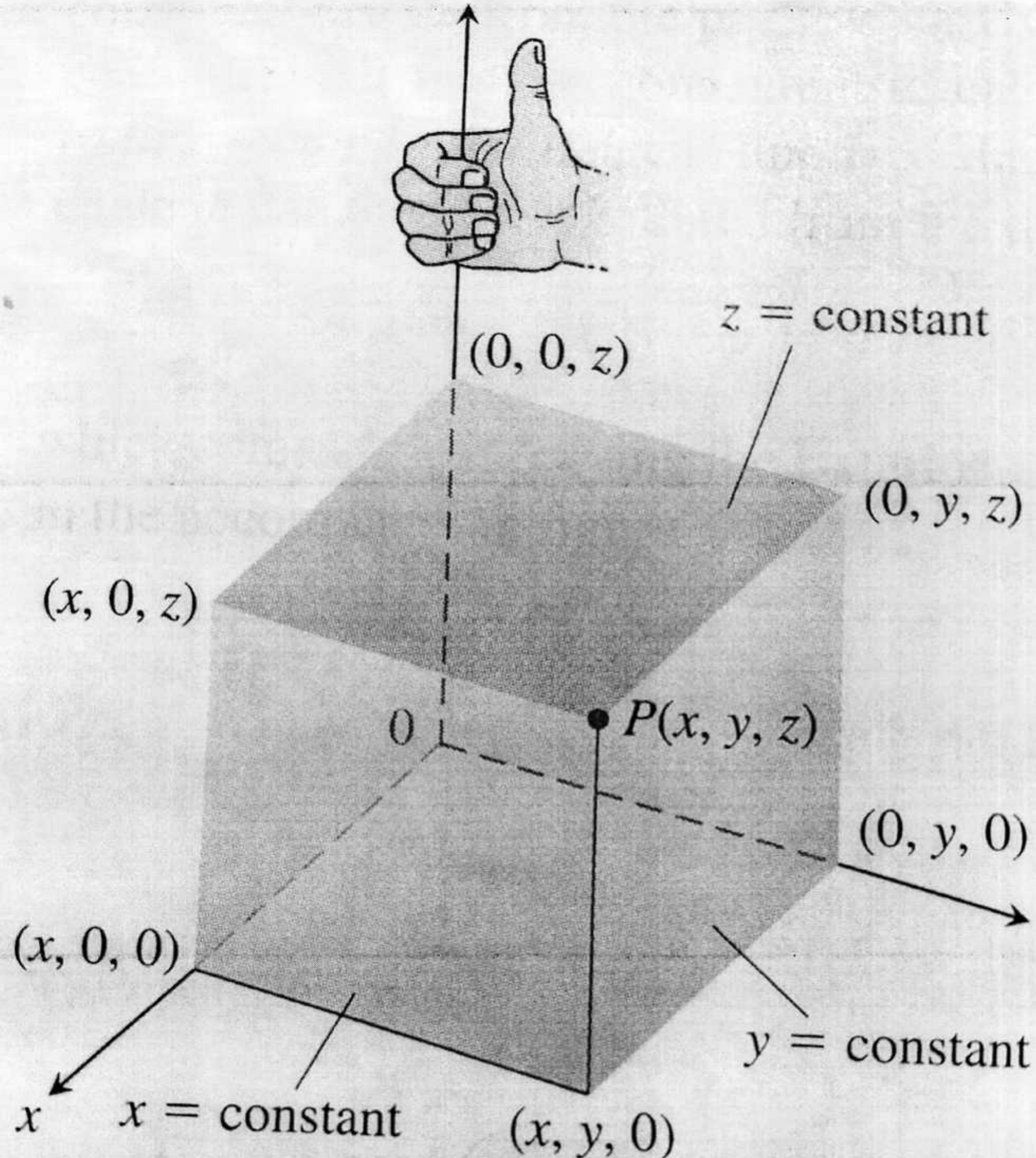
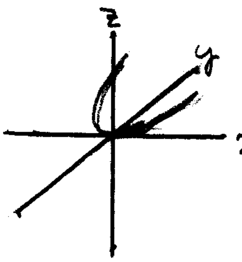
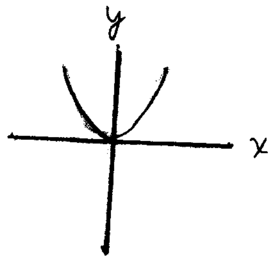
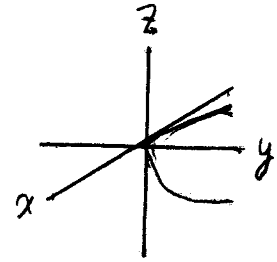


FIGURE 12.1 The Cartesian coordinate system is right-handed.





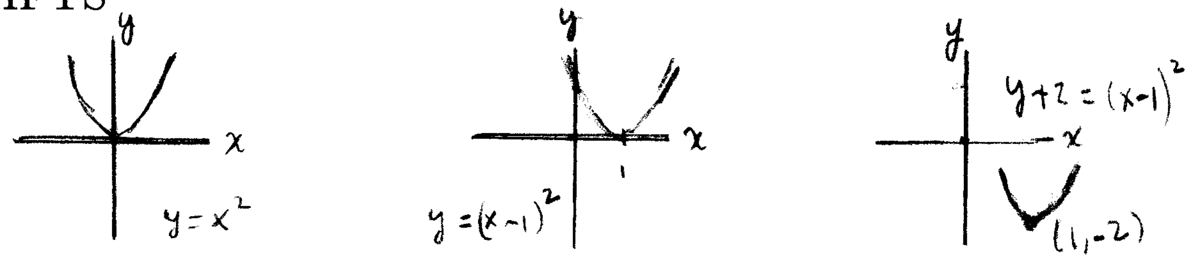


In these notes we show the x-axis in the plane of the paper, blackboard or screen on which it is drawn, as in the graph above on the left. This way, if we wish to look at the horizontal $xy-$plane turned up as in the diagram above in the middle, which shows a parabola drawn in the $xy-$plane, we only have to visualize the $xy-$plane being rotated about the $x-$axis. From the point of view of most texts, as shown above on the right, the $xy-$plane has to be rotated first about the $z-$axis and then about the $x-$axis.

Just like the coordinate axes in the plane divide the plane into four quarters called quadrants, the coordinate planes divide 3-dimensional space into eight *octants*. The octant consisting of all points with all positive coordinates is called the *first* octant. From our perspective, the first octant is in the top right half of space, behind the screen. No other octant has a standard name.

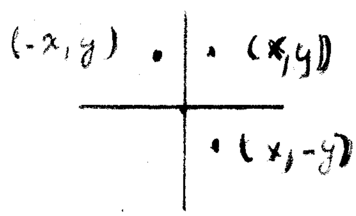
The graph of an equation in x, y, and z is the set of all points (a, b, c) such that when all occurrences of x, y, z are replaced by a, b, c, respectively, the resulting statement is true. For example, the graph of $z = 1$ is a horizontal plane one unit above the $xy-$plane, and the graph of $x^2 + y^2 + (z-1)^2 = 0$ is the single point $(0, 0, 1)$.

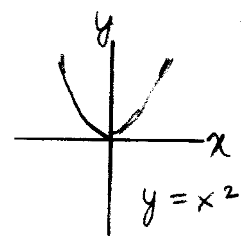
SHIFTS



Recall that, in an equation in x and y, replacing all occurences of x (y) by $x - a$ ($y - b$) shifts (also referred to as translates) the graph of the equation horizontally by a (verticaly by b). Examples of these shifts are shown in the diagrams above. Graphs in three dimensions are shifted in exactly the same way.

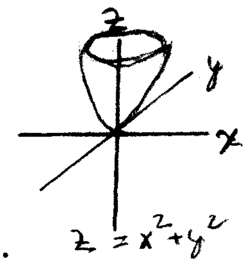
REFLECTIONS AND SYMMETRY





Recall that the reflection of the point (x, y) about the x-axis [y-axis] is $(x, -y)$ [$(-x, y)$]. That is, the variable we replace by its negative is the opposite of the axis variable. Therefore, the graph of an equation in x and y has symmetry about an axis, replacing the opposite variable by its negative yields an equivalent equation (i.e., one with the same graph). For example for the graph of $y = x^2$, the fact that $y = (-x)^2$ has the same graph as the original equation shows that the graph of $y = x^2$ has symmetry about the y-axis. By looking at $-y = x^2$, we see the original graph does not have symmetry about the x-axis. Similarly, inversion (or reflection) of (x, y) about the origin is $(-x, -y)$, so the test for an equation being symmetric about the origin is to replace both x and y by their negatives. The graph has symmetry about the origin if the new equation is equivalent to the original: $-y = (-x)^2$ is not equivalent to the original, the graph of $y = x^2$ is not symmetric about the origin.

These tests have analogues for graphs of equations in three dimensions. In this setting we can talk about symmetry about a coordinate plane, an axis, and the origin.



For example, to test for symmetry about the

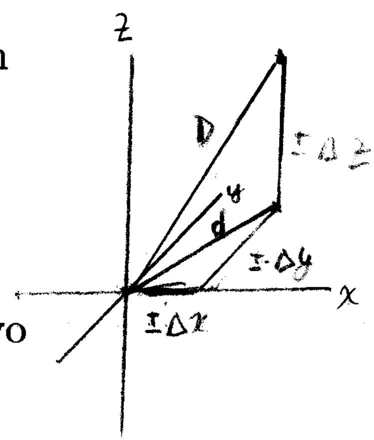
$xy-$plane: replace z by its negative;
z-axis: replace x and y by their negatives;
origin: replace all three variables by their negatives.

In each case we replace the variables not involved in describing the type of symmetry by their negatives. We will show later that the graph of $z = x^2 + y^2$ is a bowl with parabolic side walls and a circular top, as shown in the graph above. You should check that it has symmetry about $xz-$ and $yz-$ planes and about the $z-$axis only.

OTHER BASIC CONSIDERATIONS

DISTANCE FORMULA

The shadow of a point (x, y, z) when light traveling perpendicular to the xy–plane shines onto the point is called the *projection* (more precisely the orthogonal projection) onto the xy–plane. Sometimes we refer to this informally as the shadow of the point in the noon day sun. If we think of the point moving vertically towards its shadow, only z is changing, so coordinates of the projection are $(x, y, 0)$.



The formula for the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is a natural analogue of the two dimensional version: $d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

The diagram at the right contains two right triangles. One is the triangle in the xy–plane with sides $\pm\Delta x$, $\pm\Delta y$ and $\pm\Delta d$.
The distance formula in the plane comes from applying the Pythagorean theorm to this triangle. The second triangle in the diagram is the one that rises vertically and has sides D, d and $\pm\Delta z$. Using the Pythagorean theorem again, we obtain $D^2 = d^2 + (\Delta z)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$, yielding the natural analogue to the two dimensional distance formula.

MIDPOINT FORMULA

From the fact that corresponding sides of similar triangles are in proportion, we can obtain another natural analogue: The midpoint of the line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}. \frac{z_1+z_2}{2})$.

EQUATION FOR A SPHERE

We use the distance formula to find an equation whose graph is a sphere with radius r centered at (h, k, l). By definition of a sphere, any point (x, y, z) on the sphere has distance r to (h, k, l): $(x-h)^2+(y-k)^2+(z-l)^2=r^2$, and this is the standard form of the equation for a sphere.

Example 1: Show graph of the following equation is a sphere, and find its center and radius:
$x^2+y^2+z^2+3x-4z+1=0$

Solution: We complete the squares to put the equation in standard form.

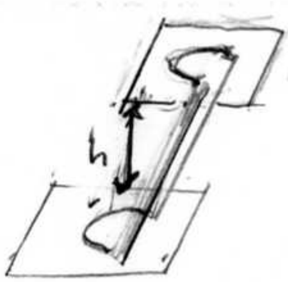
$$(x+\frac{3}{2})^2-\frac{9}{4}+y^2+(z-2)^2-4+1=0$$

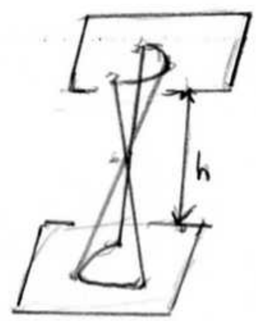
$$(x+\frac{3}{2})^2+y^2+(z-2)^2=\frac{9}{4}+4-1=\frac{21}{4},$$

so the center is $(-\frac{3}{2}, 0, 2)$ and the radius is $\sqrt{\frac{21}{4}}=\frac{\sqrt{21}}{2}$

CYLINDERS (AND CONES)

We recall the general definition of a cylinder. Given some curve in a plane in space and a direction, which now will be represented by a line, the cylinder consists of all lines through each point in the curve in the given direction, i.e., parallel to the line representing the direction. Another way to phrase this is to sweep the curve parallel to the line in the given direction indefinitely far either way to get a surface. See the example on the left below. The term cylinder may also be used for the portion of the cylinder as defined above which is between two parallel planes, with the distance between those planes called the *height* of the cylinder.





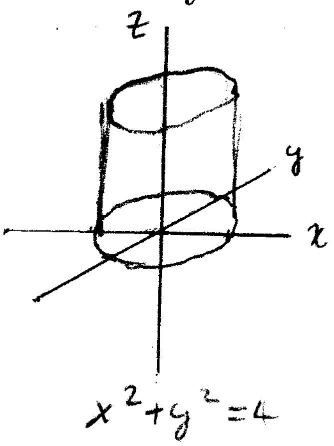
Given a curve in some plane in space and a point V not in the plane of the curve, the cone consists of all lines through V and a point on the curve. See the diagram below on the right. The portion of the cone between the vertex to the curve is a also referred to as a cone.

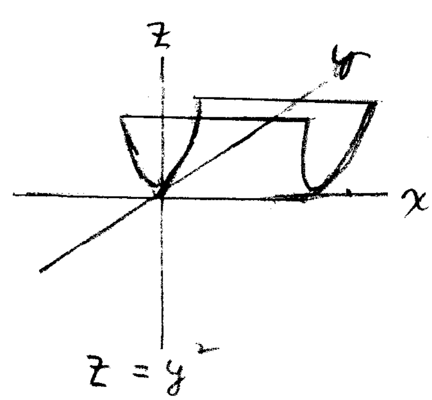
Our interest now is in the cylinders that arise from graphs of equations in two of the variables x, y and z that are graphed in three dimensions.

Example 2: Graph $x^2 + y^2 = 4$ in $\mathbf{R}^3$.

Solution: If we look at points in the $xy-$plane of R^3, namely points of form $(x, y, 0)$, then the portion of the graph in the $xy-$plane is the circle of radius 2 centered at the origin. If we take a point and move it vertically, the $x-$ and $y-$ coordinates won't change. Thus, if we take any point on the circle and move it vertically, the moved point will still satisfy $x^2 + y^2 = 4$, so any point vertically above or below the circle in the plane will also be part of the graph. I.e., the cylinder swept out by moving the circle straight up and down is the graph. A cylinder which is obtained by sweeping a curve in a direction perpendicular to the plane of the curve is called a *right* cylinder. In particular, since the curve here is a circle that is being swept perpendicular to its plane, we call this a right circular cylinder.

For any equation in two variables graphed in $\mathbf{R}^3$, we graph the figure in the two dimensions of the variables in the equation and sweep that curve perpendicularly in the direction of the missing variable. Below is the cylinder of Example 2 and the cylinder $z = y^2$, whose graph could be viewed as a trough, indefinitely wide and indefinitely tall.





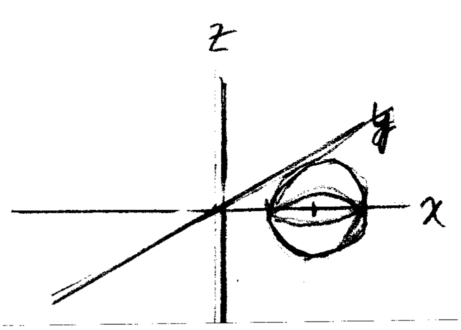
INTERCEPTS

For the graph of an equation in the plane, we find its $x-$ [$y-$]intercepts by setting the opposite variable y [x] equal to zero. In three dimensions, one can look for, in addition to $x-$, $y-$ and $z-$axis intercepts, where a graph intersects the $xy-$, $xz-$ and $yz-$planes. These intercepts can be found in a manner that generalizes the "set the opposite variable equal to zero" method in $\mathbf{R}^2$. Here is how we find some intercepts:

$x-$intercepts: set $y = 0$, $z = 0$
$xy-$intercepts: set $z = 0$

That is, set all variables not involved in describing the intercept equal to zero.

Example 3: Describe the intercepts of the equation $(x-2)^2 + y^2 + z^2 = 1$.
Solution: $x-$intercepts: $(x-2)^2 = 1$; $x = 1, 3$
$y-$intecepts: $(-2)^2 + y^2 = 1$; $y^2 = -3$; no $y-$intercepts
similarly no $z-$intercepts
$xy-$intercepts: $(x-2)^2 + y^2 = 1$ a circle centered at $(2, 0)$
$xz-$intercepts: similarly a circle in the $xz-$plane
$yz-$intercepts: $(-2)^2 + y^2 + z^2 = 1$; $y^2 + z^2 = -3$; none



The set of points at which a surface and a plane intersect is called the *trace* (or *cross-section*) of the surface on the plane. Thus, the set of xy-intercepts would usually be referred to as the trace on the xy-plane.

Section 12.6

Conic Sections and Quadric Surfaces

CONIC SECTIONS

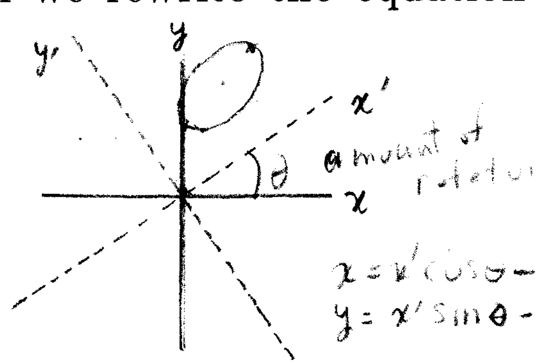
A *conic section* is the intersection of a right circular cone and a plane.

THE GENERAL SECOND DEGREE EQUATION

The degree of a term of a polynomial in more than one variable is the sum of the exponents of the variables in the term. For example, the term $2x^3yz^2$ has degree $3+1+2=6$. By the general second degree polynomial in two variables, we mean all polynomials in two variables whose terms have degree at most two and at least one term is of degree two. The general second degree equation is the equation that a general second degree polynomial equals zero, i.e., it is of the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0.$$

We ask the question, seemingly unrelated to conic sections, of what are all possible graphs of second degree polynomial equations. To answer this question we rewrite the equation several different ways.



It is possible to express in a rotated coordinate system, any graph of a second degree polynomial so that the coefficient C of the xy term is zero. Thus, we will only consider second degree equations with no xy term.

Conic Sections

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St. Vincent
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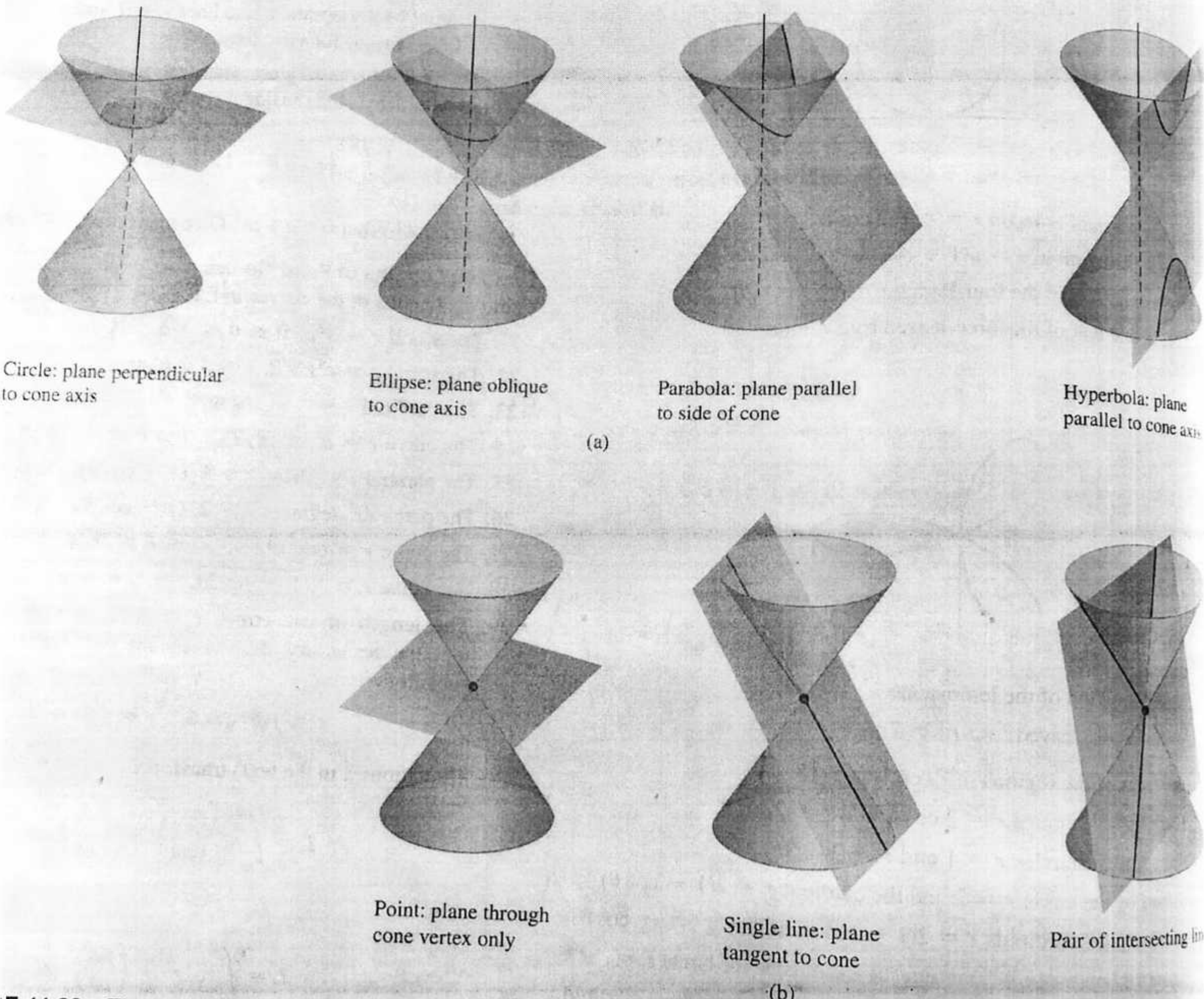
In this section we define and review parabolas, ellipses, and hyperbolas geometrically an derive their standard Cartesian equations. These curves are called *conic sections* or *coni* because they are formed by cutting a double cone with a plane (Figure 11.39). Th



E 11.39 The standard conic sections (a) are the curves in which a plane cuts a *double* cone. Hyperbolas come in two parts, *ranches*. The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

If both $A \neq 0$ and $D \neq 0$, then we can complete the square and write

$$Ax^2 + Dx + F = A(x - \alpha)^2 + F'.$$

Similarly we can complete the square if $B \neq 0$ and $E \neq 0$ to get $B(y-\beta)^2 + F'$. Since α and β are shifts, we can consider only the case for which there is square term or a linear term but not both, for both variables, and all possible graphs of the general second degree equation are rotations and/or shifts of equations of the form

$$\begin{cases} Ax^2 \\ Ax \end{cases} + \begin{cases} By^2 \\ By \end{cases} = F,$$

where we pick just one term in each bracketed pair. If $F \neq 0$, we can transpose the constant term to the right and divide both sides of the equation by the transposed constant to obtain the form

$$\begin{cases} Ax^2 \\ Ax \end{cases} + \begin{cases} By^2 \\ By \end{cases} = \begin{cases} 1 \\ 0 \end{cases}$$

For example,

$$\begin{aligned} 2x^2 + 0y^2 + 4x + y + 6 &= 2[x^2 - 2x] + y + 6 \\ &= 2[(x-1)^2 - 1] + y + 6 \\ &= 2(x-1)^2 + y + (-2+6) = 0 \end{aligned}$$

If we can graph $-\frac{1}{2}x^2 - \frac{1}{4}y = 1$, then we just shift one unit horizontally to obtain the graph of the equation displayed above.

We make one last modification: If $A > 0$, then we may write A in the form $A = \frac{1}{a^2}$ (let $a = 1/\sqrt{A}$). If $A < 0$, write $A = -\frac{1}{a^2}$, where $a = 1/\sqrt{|A|}$ Similarly write $B = \pm\frac{1}{b^2}$, and make this change of notation on the top lines above:

$$\begin{cases} \pm \dfrac{x^2}{a^2} \\ Ax \end{cases} + \begin{cases} \pm \dfrac{y^2}{b^2} \\ By \end{cases} = \begin{cases} 1 \\ 0 \end{cases}$$

We now examine each possibility with last forms written above:

Case 1: The constant is 1.

(a) Both terms on the left are squares

(i) 2 pluses

(ii) 1 plus

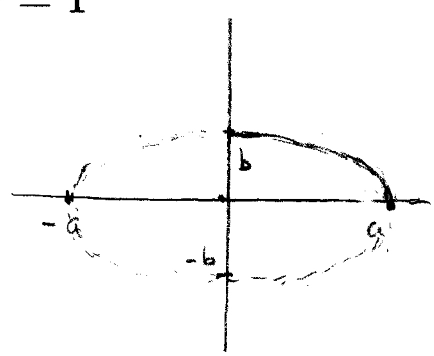
(iii) both minuses

(b) There is only one square term.

(c) No square terms: then equation is linear, not second degree.

Case 1(a)(i)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



We construct the graph. First we find the intercepts. Set $y = 0$ to find the x intercepts: $x^2 = a^2$ and $x = \pm a$. Similarly $y = \pm b$. We graph the portion of the graph in the first quadrant, by solving for y. The first and second derivatives are both negative, so the curve is decreasing and concave down. Finally we see, by replacing first x and then y by their negatives, that the graph has symmetry about both the $x-$ and $y-$axes. This graph is called an *ellipse*. When $a = b$ we get the circle of radius a as a special case.

For use in our discussion of quadric surfaces, we need to look at a slightly more general equation that that given for the ellipse.

Example: For $d = \frac{1}{2}, 1, 4$ graph the equation

$$\frac{x^2}{4} + \frac{y^2}{9} = d$$

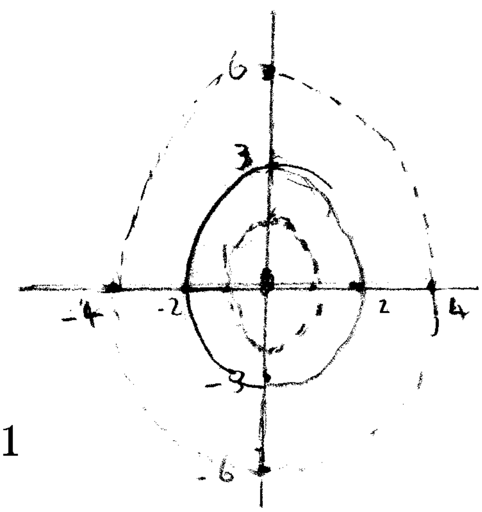
Solution: For $d = 1$ this is the ellipse with vertices $(\pm 2, 0)$ and $(0, \pm 3)$.

For $d = 4$, we write

$$\frac{x^2}{4} + \frac{y^2}{9} = 4$$

$$\frac{x^2}{4 \cdot 4} + \frac{y^2}{4 \cdot 9} = 1$$

$$\frac{x^2}{(\sqrt{4} \cdot 2)^2} + \frac{y^2}{(\sqrt{4} \cdot 3)^2} = 1$$



This an ellipse with parameters a and b that are twice those of the equation with $d = 1$.

For $d = \frac{1}{2}$

$$\frac{x^2}{4} + \frac{y^2}{9} = \frac{1}{2}$$

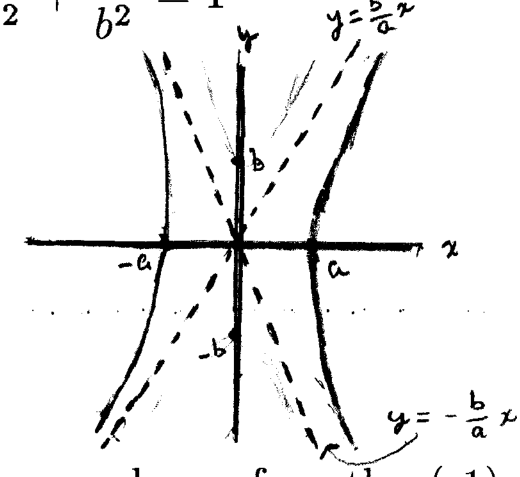
$$\frac{x^2}{\frac{1}{2} 4} + \frac{y^2}{\frac{1}{2} \cdot 9} = 1,$$

and the parameters here are $\sqrt{\frac{1}{2}}$ times the parameters with $d = 1$.

The conclusion is that the parameters for d are $\sqrt{d}$ times that parameters when $d = 1$.

Case 1(a)(ii)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



We graph the first equation, beginning by finding the intercepts. For $x-$intercepts, $x^2 = a^2$, so $x = \pm a$, For $y-$intercepts $-y^2 = a^2$. Since $-y^2 \leq 0$ and $a^2 > 0$, there are no $y-$intercepts. In the first quadrant $y = b\sqrt{(x^2/a^2) - 1}$. For large values of x, the (-1) term in the radical is very small compared to x^2/a^2, so $y = \pm\frac{b}{a}x$ are asymptotes. Since $y' > 0$ and $y'' < 0$ in the first quadrant, the curve is increasing and concave down. Again, symmetry about both axes, allows us to finish the graph. This graph is called a *hyperbola*.

For the second equation, there will be intercepts $y = \pm b$ and no $x-$intercepts, and the graph will open up and down, instead of left and right.

Case1(a) (iii)

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since the left side is less than or equal to zero for all x and y, the graph is empty.

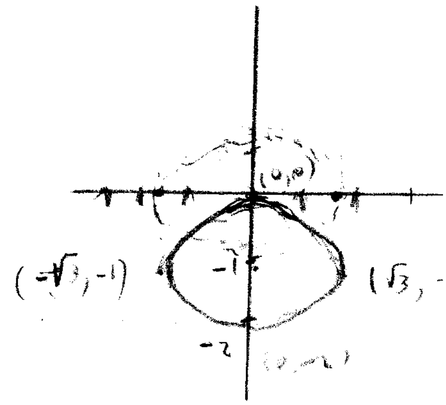
Case 1(b) $x^2/a^2 + By = 1$ or $Ax + y^2/b^2 = 1$ is clearly a parabola opening up or down for the first equation and left or right for the second case.

Case 2, the constant is 0, yields only two degenerate graphs and a duplication of Case 1(b). Thus, the non-degenerate graphs of all second degree equations coincide with the non-degenerate conic sections.

Example 1: Graph $x^2 + 3y^2 + 6y = 0$

Solution: Complete the square:

$$x^2 + 3[y^2 + 2y] = 0$$
$$x^2 + 3[(y+1)^2 - 1] = 0$$
$$x^2 + 3(y+1)^2 - 3 = 0$$
$$x^2 + 3(y+1)^2 = 3$$
$$\frac{x^2}{3} + (y+1)^2 = 1$$
$$\frac{x^2}{(\sqrt{3})^2} + \frac{(y+1)^2}{1} = 1$$



QUADRIC SURFACES

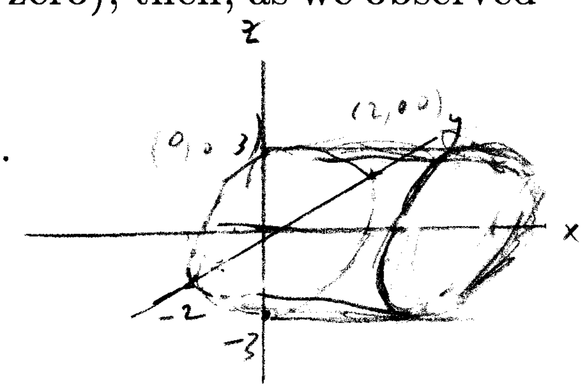
A *quadric surface* is a graph of the general second degree polynomial equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

We first observe that if only two of the variables appear in the second degree equation (i.e., both the square and linear term of one variable have coefficient zero), then, as we observed earlier, the graph is a cylinder.

Example 2: $\frac{y^2}{4} + \frac{z^2}{9} = 1$ in $\mathbf{R}^3$.

Solution:



Example 3: Describe the graph of $(x-1)(x-3) = 0$ in $\mathbf{R}^3$

Solution: It is the two planes $x = 1$ and $x = 3$ which are perpendicular to the $x-$axis. In general the graph of a second degree equation in just one variable is the empty set or one or two planes perpendicular to the axis of the variable that appears in the equation.

We wish to catalogue quadric surfaces in a manner analogous to what we did for conic sections. Since our discussion shows what is possible if only one or two of the variables appear with nonzero coefficients, we assume all three variables appear in the equation.

Two rotations (not requiring advanced mathematical techniques, but intricate enough that it never would be included in an elementary calculus book) will eliminate the cross product terms by describing the graph in the rotated coordinate axes.

Also, as for conics, we can complete the squares and shift to assume our equations are of the form

$$\begin{cases} \pm \dfrac{x^2}{a^2} \\ Ax \end{cases} + \begin{cases} \pm \dfrac{y^2}{b^2} \\ By \end{cases} + \begin{cases} \pm \dfrac{z^2}{c^2} \\ Cz \end{cases} = \begin{cases} 1 \\ 0 \end{cases}$$

Case 1: The constant is 1.

(a) 3 terms on the left are squares
- (i) 3 pluses
- (ii) 2 pluses
- (iii) 1 plus
- (iv) no pluses: left side less than 0, so empty graph

(b) 2 terms on the left are squares
- (i) no pluses or 2 pluses
- (ii) 1 plus

(c) 1 term on the left is a square

(d) No square terms: then equation is linear, not second degree.

Case 2: The constant is 0.

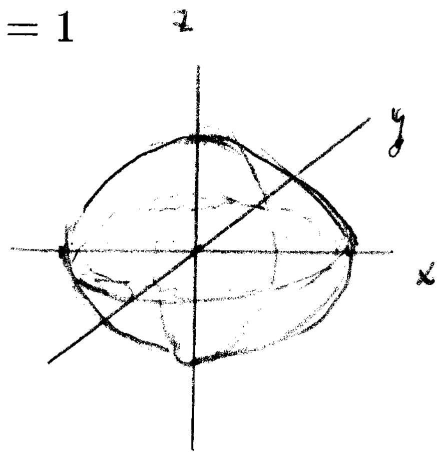
(a) 3 terms on the left are squares
- (i) 3 pluses or no pluses: graph is just the origin
- (ii) 1 or 2 pluses

(b) 2 terms on the left are squares: same as Case 1(b)

(c) 1 term on the left is a square: same as Case 1(c)

Case 1(a)(i)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



We think of how a clay pot may be built in layers. Put a frame up and build the layers around that. For our "frame" we look at the portion of the graph that is in the the plane of the screen, the xz−plane from my perspective, that is, the portion of the graph with $y = 0$:

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1,$$

which is an ellipse, as shown at the right. Then we build horizontal strips at height $z = z_0$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z_0^2}{c^2} = d$$

This is an ellipse with dimension shrunk in proportion by a factor $\sqrt{d}$. By symmetry the bottom half is symmetric with the top half. This figure is called an *ellipsoid*.