MATH 212 SAMPLE FINAL EXAM SOLUTIONS

1. (a) Evaluate: $\int_1^2 \frac{x^2+x+1}{x^3+x^2}\,dx$

(b) Evaluate: $\int e^x \sin x\,dx$

(a) $\int_1^2 \frac{x^2+x+1}{x^2(x+1)}\,dx =$

$$\int_1^2 \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$= \int_1^2 \frac{1}{x^2} + \frac{1}{x+1}\,dx$$

$$= -\frac{1}{x}\bigg|_1^2 + \ln|x+1|\big|_1^2$$

$$= \frac{1}{2} + \ln 3 - \ln 2$$

$$= \frac{1}{2} + \ln\frac{3}{2}.$$

$x^2 + x + 1 =$

$A(x^2 + x) + B(x + 1) + Cx^2$

$(A + C)x^2 + (A + B)x + B$

$B = 1$

$A + B = 1 \Rightarrow A = 0$

$A + C = 1 \Rightarrow C = 1$

(b) Let $I = \int e^x \sin x\,dx$, and integrate by parts with e^x being the factor integrated:

$$\begin{aligned}
I &= e^x \sin x - \int e^x \cos x\,dx \\
&= e^x \sin x - \left(e^x \cos x - \int e^x(-\sin x)\,dx\right) \\
&= e^x \sin x - e^x \cos x - I \\
2I &= (\sin x - \cos x)e^x \\
I &= \frac{(\sin x - \cos x)e^x}{2}.
\end{aligned}$$

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2. (a) Evaluate: $\int \sin^2(3x)\cos^3(3x)\,dx$

(b) Evaluate: $\int_0^{\pi/6} \cos^2(3x)\,dx$

(c) Find the Trapezoidal Rule estimate of $\int_0^2 x^4\,dx$ obtained by breaking the interval of integration into $n = 4$ subintervals.

(a) Let $u = \sin(3x)$. Then $du = 3\cos(3x)\,dx$ and

$$\begin{aligned}
&\int \sin^2(3x)\cos^3(3x)\,dx \\
&= \int \sin^2(3x)[1 - \sin^2(3x)]\cos(3x)\,dx \\
&= \int u^2(1 - u^2)\frac{1}{3}\,du \\
&= \frac{1}{3}\left(\frac{u^3}{3} - \frac{u^5}{5}\right) = \frac{\sin^3(3x)}{9} - \frac{\sin^5(3x)}{15}.
\end{aligned}$$

(b)
$$\begin{aligned}
\int_0^{\pi/6} \cos^2(3x)\,dx &= \int_0^{\pi/6} \frac{1}{2} + \frac{1}{2} + \cos(6x)\,dx \\
&= \frac{\pi}{12} + \frac{1}{2}\left.\left(\frac{1}{6}\sin(6x)\right)\right|_0^{\pi} 6 = \frac{\pi}{12}.
\end{aligned}$$

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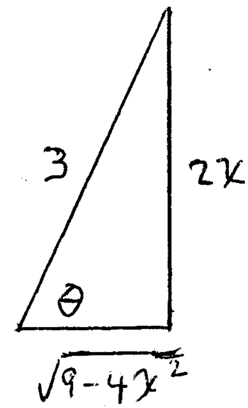
3. (a) Evaluate: $\int \frac{x^3}{\sqrt{9-4x^2}}\,dx$

(a) We will make two substitutions: First, $x = \frac{3}{2}\sin\theta$, so $dx = \frac{3}{2}\cos\theta\,dx$ and $\sin\theta = \frac{2x}{3}$. Second, $u = \cos\theta$ and

$du = -\sin\theta\, d\theta.$

$$\int \frac{x^3\,dx}{\sqrt{9-4x^2}} = \frac{27}{8}\int \frac{\sin^3\theta}{3\cos\theta}\left(\frac{3}{2}\cos\theta\right)\,d\theta$$
$$= \int \frac{27}{16}\sin^3\theta\,d\theta = \frac{27}{16}\int (1-\cos^2\theta)\sin\theta\,d\theta$$
$$= \frac{27}{16}\int u^2 - 1\,du = \frac{27}{16}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)$$
$$= \frac{27}{16}\left[\frac{(9-4x^2)^{3/2}}{3\cdot 3^3} - \frac{(9-4x^2)^{1/2}}{3}\right]$$



This is an appropriate form for the answer, but it can be simplified further by factoring:

$$= \frac{27\sqrt{9-4x^2}}{16\cdot 3}\left[\frac{9-4x^2}{27} - 1\right] = \frac{9}{16}\left[\frac{-18-4x^2}{27}\right]\sqrt{9-4x^2}$$
$$= \frac{9}{16}\left(\frac{-2}{27}\right)(9+2x^2)\sqrt{9-4x^2} = -\frac{1}{24}(9+2x^2)\sqrt{9-4x^2}$$

(b) Find the value of each convergent improper integral and show why each divergent improper integral is divergent.

(i) $\int_2^\infty \frac{2+e^{-x}}{x+2}\,dx$ (ii) $\int_2^4 \frac{1}{(x-2)^2}\,dx$ (iii) $\int_2^\infty e^{2-x}\,dx$

(i) Substitute $u = x+2$; then $du = dx$, $u(2) = 4$ and $u(\infty) = \infty$, to a obtain divergent integral in u, so that the original integral is divergent.

$$\int_2^\infty \frac{2+e^{-x}}{x+2}\,dx \geq \int_2^\infty \frac{2}{x+2}\,dx = \int_4^\infty \frac{2}{u}\,du.$$

(ii) Again we use a substitution, $u = x-2$, and note $\int_0^2 \frac{1}{u^p}\,du$ is divergent for $p > 1$:

$$\int_2^4 \frac{1}{(x-2)^2}\,dx = \int_0^2 \frac{1}{u^2}\,du.$$

(iii) We evaluate the integral to show it is convergent:

$$\int_2^\infty e^{2-x}\,dx = e^2 \int_2^\infty e^{-x}\,dx$$
$$= -e^2\, e^{-x}\Big|_2^\infty = e^2\, e^{-x}\Big|_\infty^2 = e^2(e^{-2} - 0) = 1.$$

Note: The variation of (i) obtained by replacing $x+2$ by $x-2$ is an example of an integral that is of Type 2 at the lower limit and Type 1 at the upper limit and thus must be written as a sum of two integrals to show that the original integral is divergent. Again we use a substitution, $u = x - 2$. Since at least one (in fact, both) of the terms of the sum is divergent, the original integral is divergent.

$$\int_2^\infty \frac{2+e^{-x}}{x-2}\,dx \geq \int_0^\infty \frac{2}{x-2}\,dx$$
$$= \int_0^\infty \frac{2}{u}\,du = \int_0^1 \frac{2}{u}\,du + \int_1^\infty \frac{2}{u}\,du.$$

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4. State, for each series, whether it converges absolutely, converges conditionally or diverges. Name a test which supports each conclusion and show the work to apply the test.

(a) $\sum_{n=0}^\infty \frac{(-1)^n n}{3n+1}$ (b) $\sum_{n=0}^\infty \frac{(-1)^n 5^n}{3^{2n}}$ (c) $\sum_{n=0}^\infty \frac{(-1)^n}{3n+1}$

For each of the series, a_n will denote the n-th term of the series.

(a) Since $|a_n| = n/(3n+1)$ converges to $1/3 \neq 0$, the series is divergent by the Test for Divergence.

(b) Since, for $\sum_0^\infty (-1)^n 5^n/3^{2n}$,

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}5^{n+1}}{3^{2n+2}} \frac{3^{2n}}{(-1)^n 5^n} = \frac{-5}{9}$$

is constant, (b) is a geometric series with common ratio that has absolute value less than 1, so the series is convergent. We could also have used the ratio test, since, the same calculation shows $|a_{n+1}/a_n|$ converges to 5/9.

(c) By the alternating series test, the series $\sum_0^\infty (-1)^n/(3n+1)$ is convergent. Using the limit comparison test, comparing $\sum_1^\infty |a_n|$ to the series $\sum_1^\infty 1/n$, we see $\sum_1^\infty |a_n|$ is divergent, so (c) is conditionally convergent.

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5. (a) Find the interval of convergence of the series $\sum_{n=0}^\infty \frac{(x-2)^n}{(n+2)3^n}$.

Remember to check the endpoints, if applicable.

(a) The center and n-th coefficient are, respectively

$$x_0 = 2 \qquad \text{and} \qquad c_n = \frac{1}{(n+2)3^n}.$$

$L_0 = \lim \left|\frac{c_{n+1}}{c_n}\right| = \lim \frac{(n+2)3^n}{(n+3)3^{n+1}} = \frac{1}{3}.$

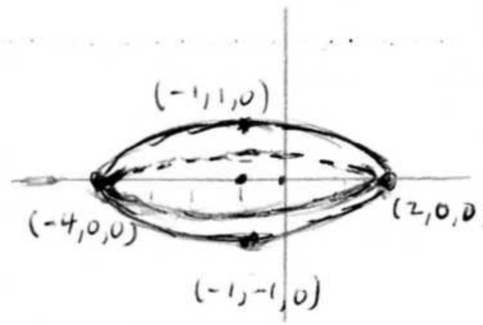
Then the radius, R, of convergence is $R = 1/L = 3$, and the series will converge exactly in the interval $(x_0 - R, x_0 + R) = (2-3, 2+3) = (-1, 5)$, together with, possibly, the two endpoints, which must be checked separately to obtain $[-1, 5)$:

$$x = -1 : \sum \frac{(-1-2)^n}{(n+2)3^n} = \sum \frac{(-1)^n}{(n+2)}, \text{ convergent, AST}$$

$$x = 5 : \sum \frac{(5-2)^n}{(n+2)3^n} = \sum \frac{1}{n+2}, \text{ divergent, } p\text{-series}$$

(b) Graph the equation $x^2 + 2x + 9y^2 + 9z^2 = 8$, labelling the coordinates of the center and one vertex if any exist.

$$x^2 + 2x + 9y^2 + 9z^2 = 8$$
$$(x+1)^2 - 1 + 9y^2 + 9z^2 = 8$$
$$(x+1)^2 + 9y^2 + 9z^2 = 9$$
$$\frac{(x+1)^2}{9} + y^2 + z^2 = 1$$



6. (a) Let $f(x) = \dfrac{1}{1+2x}$. (i) Find the first five terms of the Maclaurin series (i.e., the series centered at 0) representation of $f(x)$.

(ii) Use the result in (i) to find $f'(.01)$ with an error less than or equal .001. Justify that your answer has the required accuracy.

$$\frac{1}{1-x} = \sum_0^\infty x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$f(x) = \frac{1}{1+2x} = \frac{1}{1-(-2x)} = \sum_0^\infty (-2x)^n$$

$$= \sum_0^\infty (-2)^n x^n = 1 - 2x + 4x^2 - 8x^3 + 16x^4 \pm \dots$$

(i) $$f'(x) = \sum_1^\infty n(-2)^n x^{n-1} = -2 + 8x - 24x^2 + 64x^3 \pm \dots$$

$$f'\left(\frac{1}{100}\right) = \sum_1^\infty n(-2)^n \left(\frac{1}{100}\right)^{n-1}$$

$|R_3| \le \frac{1}{1000}$

(ii) $$= -2 + \frac{8}{100} - \frac{24}{100^2} + \frac{64}{100^3} \pm \dots$$

(b) Find the limit or show it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^4}{x^4+y^2}$$

$$y=0: \lim_{x\to 0} \frac{x^2+0}{x^4+0} = \infty; \qquad x=0: \lim_{y\to 0} \frac{0+y^4}{0+y^2} = 0.$$

Since there are different limits along different paths, the limit of the function does not exist.

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7. (a) Find an equation of the plane containing the points $A(1,0,-1)$, $B(2,-1,0)$ and $C(1,2,3)$.

(b) Find parametric equations for the line through $A(5,8,0)$ and parallel to the line through $B(4,1,-3)$ and $C(2,0,2)$.

(c) Is the vector $\mathbf{v}$ parallel, perpendicular or neither to the plane $z = x + 2y$, where

(i) $\mathbf{v}_1 = \langle 2,0,2\rangle$ and (ii) $\mathbf{v}_2 = \langle 1,2,1\rangle$?

(a) Vectors $\mathbf{AB} = \langle 1,-1,1\rangle$ and $\mathbf{AC} = \langle 0,2,4\rangle$ are parallel to the plane, so their cross product, call it $\mathbf{n}$, is a normal:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 2 & 4 \end{vmatrix} = \langle -6,-4,2\rangle$$

Choosing $(-1/2)\mathbf{n} = \langle 3,2,-1\rangle$ as the normal and point A to substitute into the standard equation of for a normal yields the equation $3(x-1)+2y-(z+1)=0$. Any other choice of a normal to the line and a point on the line would be an equally acceptable answer.

(b) $\mathbf{CB} = \langle 2,1,-5\rangle$ is parallel line we seek, so the line is

$x = 5+2t, \qquad y = 8+t, \qquad z = -5t.$

(c) The plane $z = x + 2y$ can be written $x + 2y - z = 0$ and has normal $\mathbf{n} = \langle 1, 2, -1 \rangle$.

(i) $\mathbf{v}_1 \cdot \mathbf{n} = \langle 2, 0, 2 \rangle \cdot \langle 1, 2, -1 \rangle = 0$, so $\mathbf{v}_1$ is perpendicular to $\mathbf{n}$, and, hence parallel to the plane.

(ii) $\mathbf{v}_2 \cdot \mathbf{n} = \langle 1, 2, 1 \rangle \cdot \langle 1, 2, -1 \rangle = 4 \neq 0$, and the second and third components of $\mathbf{v}_2$ and $\mathbf{n}$ are not in proportion, so $\mathbf{v}_2$ and the plane are neither parallel or perpendicular to each other.

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8. (a) Find parametric equations for the line of intersection of the planes $2x + 3y + z = 1$ and $x - 3y + 2z = 2$.

(a) We will solve the two equations for x and y in terms of z. Adding the two equations together yields, $3x + 3z = 3$, by dividing both sides by 3 and bringing the second term to the right we obtain $x = 1 - z$. We substitute this result into the second equation and solve for y:

$(1 - z) - 3y + 2z = 2, \qquad$ or $\qquad y = -\frac{1}{3} + \frac{1}{3}z.$

to obtain parametric equations

$x = 1 - z, \qquad y = -\frac{1}{3} + \frac{1}{3}z, \qquad z = z.$

If you choose, you could let $z = 3t$ to obtain

$x = 1 - 3t, \qquad y = -\frac{1}{3} + t, \qquad z = 3t.$

Also, one could find any one point on the line of intersection and take the cross product of the two normals to obtain the direction of the line.

Yet another solution is to find two points, A and B, on the line and take $\mathbf{AB}$ as a vector in the direction of the line.

Finally, we could have solved the varables in terms of x or y instead of z.

(b) Find the distance between the planes $x + 2y + 2z = 2$ and $x + 2y + 2z = 4$.

Since the planes have the same normal, they are parallel, so the distance between the planes is the same as the distance from any point on one plane to the other plane. We can find the point at which the z-axis intersects the first plane by substituting $x = y = 0$ and solving for z:

$0 + 0 + 2z = 2$, so $z = 1$ and $(0, 0, 1)$ is a point on the first plane.

A formula for the distance D from $S(x_0, y_0, z_0)$ to the plane with equation $ax + by + cz + d = 0$ which may be more convenient than the vector form is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1 \cdot 0 + 2 \cdot 0 + 2 \cdot 1 - 4|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{2}{3}.$$

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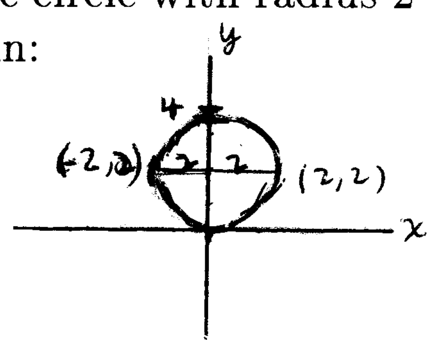
9. (a) Find the rectangular coordinates of the point(s) of the graph of the polar equation $r = 4 \sin\theta$ that are farthest from the y-axis.

(a) By multiplying both sides of the equation by r and completing the square we obtain the circle with radius 2 which is tangent to the x-axis at the origin:

$$\begin{aligned} r^2 &= 4r \sin\theta \\ x^2 + y^2 &= 4y \\ x^2 + (y-2)^2 &= 4 \end{aligned}$$



(b) Sketch the graph of the parametric equations $x = 3\cos t,\ y = 4\sin t$, labelling all intercepts.

(b) Notice that $(x(t+2\pi), y(t+2\pi)) = (x(t), y(t))$, so that all points on the graph will be obtained by letting t range from 0 to 2π.

To find the x-intercepts, set $y = 4\sin t = 0$ to obtain $t = 0, \pi$ and $x = 3, -3$. Similarly, the y-intercepts are $y = \pm 4$.

As t increases from 0 to $\pi/2$, x decreases and y increases, and as t increase from $\pi/2$ to π, x continues to decrease, and y decreases. This shows that the graph increases for $-3 \le x \le 0$ and decreases for $0 \le x \le 3$.

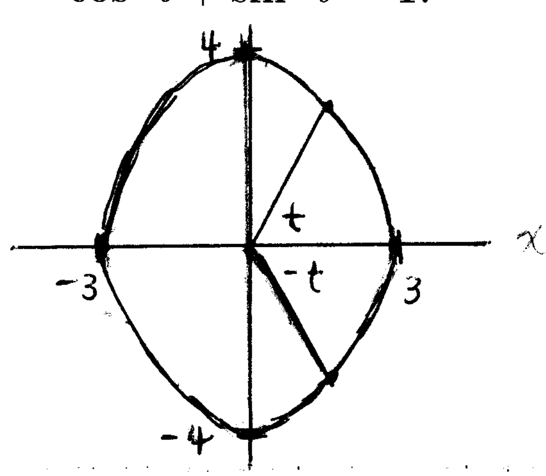
Note that $(x(-t), y(-t)) = (x(t), -y(t))$, so that the graph has symmetry about the x-axis, as illustrated below. The portion of the graph below the x-axis is the mirror image of the graph above the x-axis, producing the graph below.

A detail that is not clear from the above analysis is that the graph above (below) the x-axis is concave down (respectively, up) as shown in the graph. Although, not something that a student would be expected to think of for an exam, all points on the graph satisfy the equation

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$\frac{9\cos^2 t}{9} + \frac{16\sin^2 t}{16} = \cos^2 t + \sin^2 t = 1.$$

Thus, the graph is an ellipse.



10. (a) Find $\frac{\partial f}{\partial x}$, for $f(x,y) = \log_2(3^{x^2} + y)$

(b) For a positive integer n, find $\dfrac{\partial^n f}{\partial z^n}$, the n-th partial derivative of f with respect to z, for $f(x,y,z) = e^{x+y^2+2z}$

(c) Find f_{xy} for $f(x,y) = xy^3 + \frac{x^2+3x+1}{\sqrt{4+x^2}}$

(a) One may convert all logs and exponentials to base e using the formulas

$$a^u = e^{u \ln a} \qquad\qquad \log_b u = \frac{\ln u}{\ln b}.$$

Then

$$\begin{aligned} f(x,y) &= \frac{\ln\left(e^{(\ln 3)x^2} + y\right)}{\ln 2} \\ f_x &= \frac{[2(\ln 3)x]e^{(\ln 3)x^2}}{e^{(ln 3)x^2} + y}\,\frac{1}{\ln 2} \\ &= \frac{[2(\ln 3)x]3^{x^2}}{(3^{x^2} + y)\ln 2} \end{aligned}$$

(b) $f_z = 2f$, $f_{zz} = 2^2 f$, and continuing in this fashion, the n-th partial derivative of f is $2^n f = 2^n e^{x+y^2+2z}$. (The formal use of this reasoning is called *mathematical induction.*)

(c) If we differentiate f first with respect to x, the derivative is messy, but if we note that $f_{xy} = f_{yx}$ and differentiate first with respect to y, then $f_y = 3y^2$, so $f_{yx} = 3y^2$.