Standard basis vectors and another notation for vectors (Do NOT mix the notations!!!!!)

In n-space, we give vectors that have one component being 1 and all others zero the title *basis vector*.

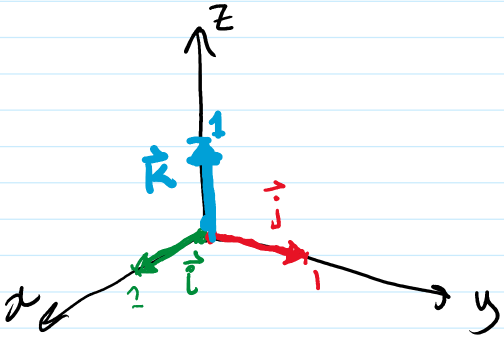
We denote these vectors by $e_i = \langle 0,0,\ldots,0,1,0,\ldots,0\rangle$, where the 1 appears in the i^{th} position.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ we call these vectors:

$\vec{\imath} = \langle 1,0\rangle$, and $\vec{\jmath} = \langle 0,1\rangle$ (in $\mathbb{R}^2$)

$\vec{\imath} = \langle 1,0,0\rangle$, $\vec{\jmath} = \langle 0,1,0\rangle$, and $\vec{k} = \langle 0,0,1\rangle$ (in $\mathbb{R}^3$)

Geometrically:



Notation-wise:

Suppose $\boxed{\vec{a} = \langle a_1, a_2, a_3\rangle}$ component notation.

$= a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle$

$$\boxed{\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}}$$

↳ ijk-notation

called a "linear combination" of $\vec{i}, \vec{j}, \vec{k}$

e.g. $\vec{a} = \langle 1, 2, -3\rangle$

is the same as

$\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$

Don't mix notations! $\cancel{\vec{a} = \langle i, 2j, -3\vec{k}\rangle}$

Unit vectors

A unit vector is simply a vector with length 1.

A unit vector is simply a vector with length 1.

Proposition: If $\vec{a} \neq \vec{0}$, then the unit vector $\vec{u}$ in the same direction as $\vec{a}$ is given by

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|}$$

Example: Find two unit vectors that are parallel to $\vec{a} = \langle 2, -1, 3 \rangle$

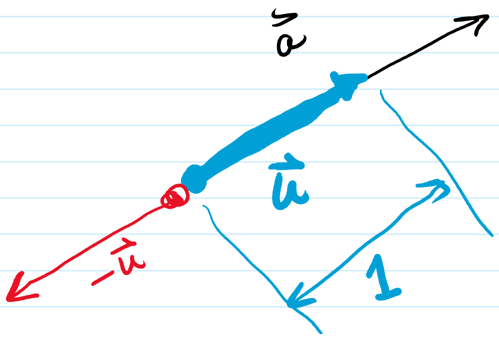
$$|\vec{a}| = \sqrt{(2)^2 + (-1)^2 + (3)^2}$$

$$= \sqrt{4+1+9}$$

$$= \sqrt{14}$$

$$\Rightarrow \quad \vec{u} = \frac{\langle 2, -1, 3 \rangle}{\sqrt{14}} = \boxed{\left\langle \frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right\rangle}$$

the other: $-\vec{u} = \boxed{\left\langle -\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}} \right\rangle}$



Math 212 GH - 4/20/2020 - The Dot Product

Wednesday, April 15, 2020 5:12 PM

12.3 - The Dot Product

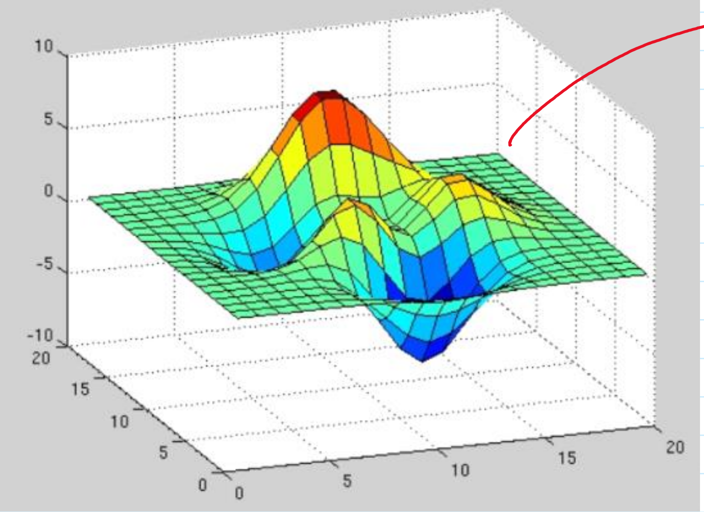
But first,

Motivation for this and the next section: Patches, directions and normal vectors

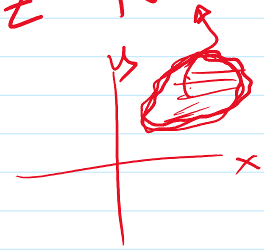
As we've seen, navigating the 3D world has its challenges, and because notation from 2D doesn't always translate well (for example, remember $x = 1$ is a line in 2D, but it is a plane in 3D), we need new ways of looking at the world.

When it comes to calculus specifically, we need new notions of dissection (how to cut things up --like how we dissected the interval $[a, b]$ into *subintervals* in order to define the integral), and direction (like how the *derivative* (slope) gave us a notion of direction (up or down, etc.).

You can't break a surface into subintervals, so we can use patches:



$z = f(x, y)$

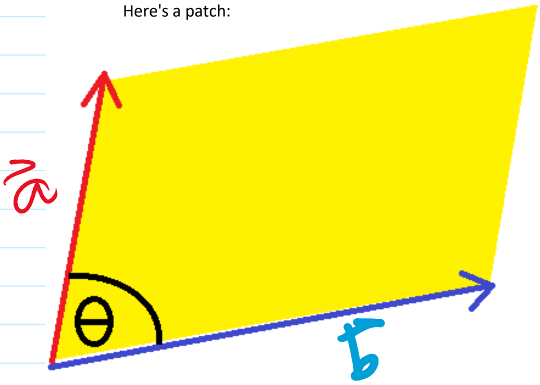


From: https://www.bu.edu/tech/support/research/training-consulting/online-tutorials/visualization-with-matlab/

(At the moment, I am going to mention certain things as being important, much without justification, but don't worry, you will soon get to a point (calculus 3) where you will see why they are important. For now, just try to extrapolate from your intuition with how things worked in 2D, single -variable calculus.)

Now, in the same way that the length of a subinterval is important, the area of these patches that we dissect a surface in will be important. And in the same way we can describe things using slope and angles in 2D, so too, having a notion of how to compute "slopes" and angles in 3D. And because vectors will be important to find position in a 3D space, we'd like to be able to somehow connect the idea of areas, angles, and direction with vectors. We do it thusly.

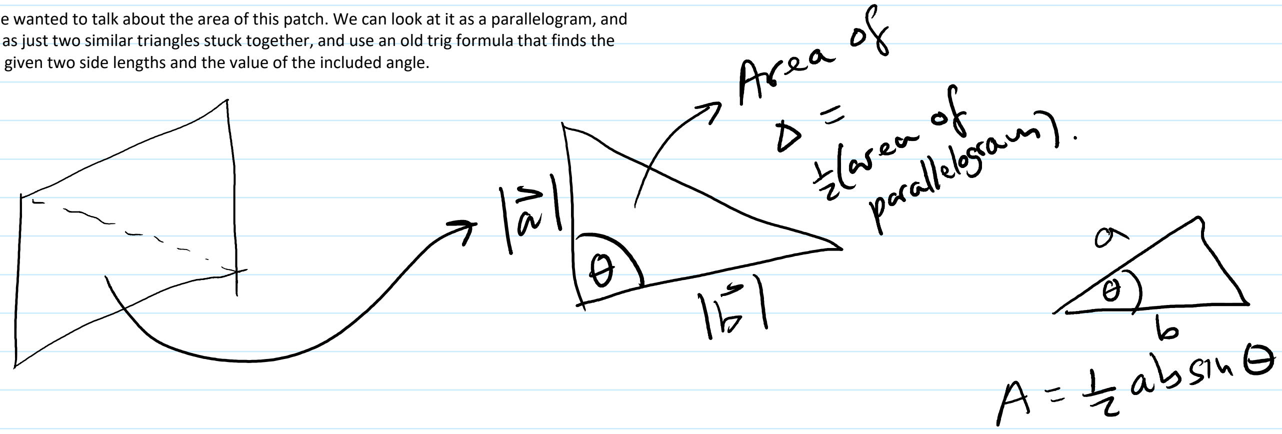
Here's a patch:



If we look at the vectors $\vec{a}$ and $\vec{b}$ along the edges of some patch, we can use some good ol' trig formulas to help us here.

For instance, if we wanted to talk about the area of this patch. We can look at it as a parallelogram, and then look at that as just two similar triangles stuck together, and use an old trig formula that finds the area of a triangle given two side lengths and the value of the included angle.

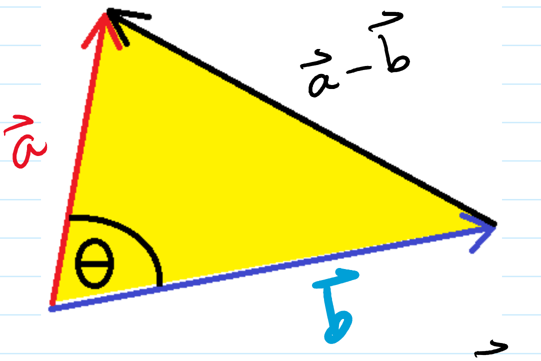
Illustration:

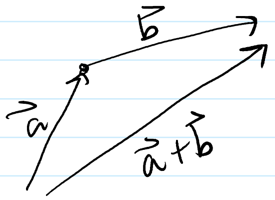


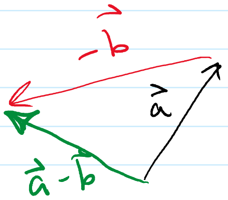
Thus, the area of this patch will be $\boxed{A = |\vec a||\vec b|\sin\theta}$. We'll come back to this in the next section!

But what about actually finding the angle? Well, again, trig comes to the rescue.

We won't need the whole parallelogram, so let's take half of it. We then get a triangle. And use some other vector to be the other side.



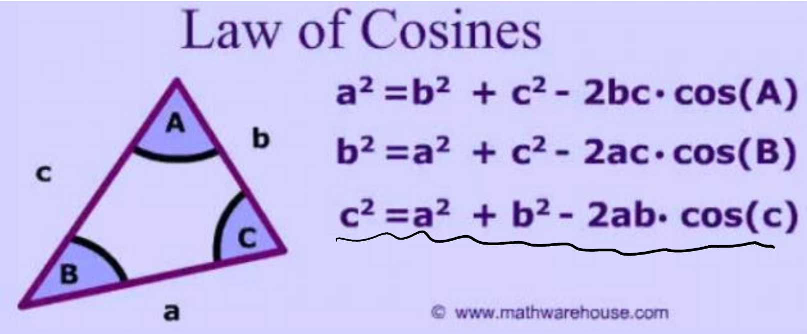




What is the vector that's on the other side (as shown in the diagram)? $\vec a - \vec b$

Now, do you remember the law of cosines from high school trig? Who am I kidding, of course you do!

But just in case...



Law of Cosines

$a^2 = b^2 + c^2 - 2bc\cdot\cos(A)$

$b^2 = a^2 + c^2 - 2ac\cdot\cos(B)$

$\underline{c^2 = a^2 + b^2 - 2ab\cdot\cos(c)}$

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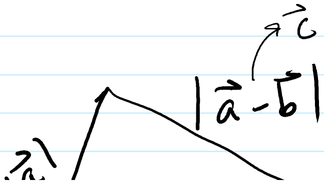
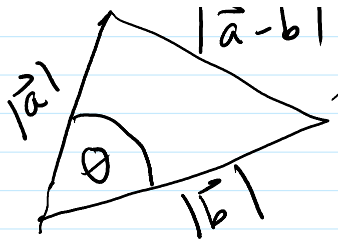
So, for our patch above, we can apply the law of cosines, the version that would include the angle θ, to obtain:



$$\boxed{|\vec a - \vec b|^2 = |\vec a|^2 + |\vec b|^2 - 2|\vec a||\vec b|\cos\theta}$$

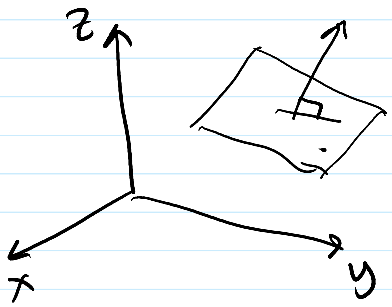


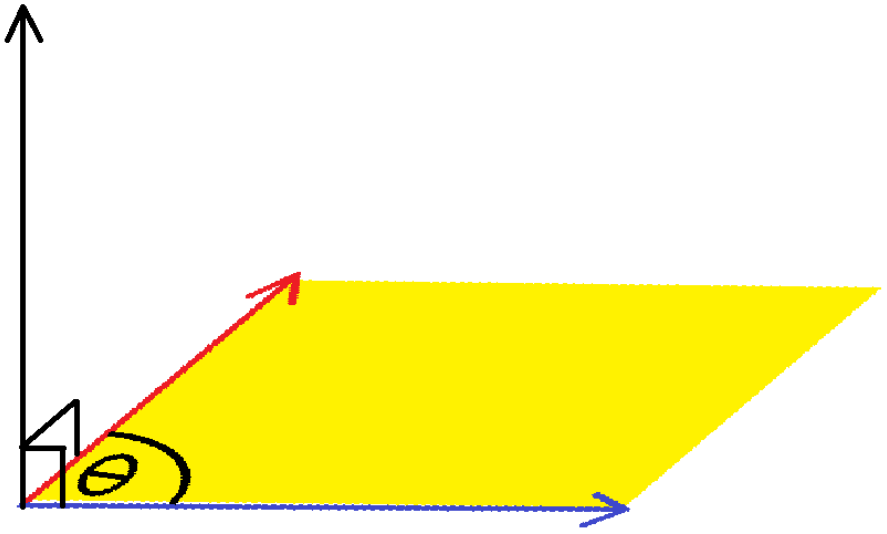
$$\xrightarrow{} \underbrace{|\vec{a}-\vec{b}| = |\vec{a}| + |\vec{b}| - 2|\vec{a}||\vec{b}|\cos\theta}$$

We'll get back to this in THIS section.

Now, for direction. How could one accurately talk about the direction in which this patch is slanted in space. In the same way we can say, "Oh, this line is in the xy-plane at a slope of 2", and everyone knows what that means, how can we talk about how the patch is sloped? Or some other meaningful notion of direction of slant?

Guesses? normal vector → a vector perpendicular to the plane/patch!





So, now we have an idea of a *normal vector*. But how can you know a vector is *normal* to a surface or a plane? Can just making sure it is perpendicular to some vector in the plane good enough? As it turns out, no! We'll illustrate.

We will, in fact, need at least *two*, non-parallel vectors in the plane, to determine a normal vector to a plane. So now, we have another need: how do we find a vector that is "perpendicular" (we'll come up with a different name for this soon) to two vectors. This is important and will be dealt with in the next section.

Now, on to the subject matter at hand!

12.3 - The Dot Product

Let's go back to the idea of finding the angle between two vectors. Let's illustrate what the calculation would look like in 2D, and hopefully you can see that this will generalize to nD, because I'm just going to state that it does... if you're not convinced, do the calcuations for 3D and 4D, and you should be convinced by then.

Suppose $\vec{a} = < a_1, a_2 >$ and $\vec{b} = < b_1, b_2 >$, we saw how we can find the angle between them.

We saw: $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

$$\Rightarrow |\langle a_1-b_1, a_2-b_2\rangle|^2 = |\langle a_1, a_2\rangle|^2 + |\langle b_1, b_2\rangle|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \cancel{a_1^2} - \cancel{2}a_1b_1 + \cancel{b_1^2} + \cancel{a_2^2} - \cancel{2}a_2b_2 + \cancel{b_2^2} = \cancel{a_1^2} + \cancel{a_2^2} + \cancel{b_1^2} + \cancel{b_2^2} - \cancel{2}|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow \boxed{\cos\theta = \frac{a_1b_1 + a_2b_2}{|\vec{a}||\vec{b}|}}$$

this is an important expression.

Now, there is an important expression in this calculation. The expression $a_1b_1 + a_2b_2$. This expression is important AND it is in a form that is easy to think of formulaically. Remember when I said I would tell you this will generalize? Well, now's that time. This generalizes to vectors in nD; if we had vectors $\vec{a} = < a_1, a_2, \ldots, a_n >$ and $\vec{b} = < b_1, b_2, \ldots, b_n >$, then the expression would be $a_1b_1 + a_2b_2 + \cdots + a_nb_n$. This screams for a definition. We'll call this expression....

The Dot Product

Defn: Let $\vec{a} = < a_1, a_2, \ldots, a_n >$ and $\vec{b} = < b_1, b_2, \ldots, b_n >$ be vectors in $\mathbb{R}^n$. Then we define the dot product of $\vec{a}$ and $\vec{b}$, called "$\vec{a}$ dot $\vec{b}$", to be the expression

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

and we denote this expression by:

$$\boxed{\vec{a}\cdot\vec{b} \;=\; a_1b_1 + a_2b_2 + \cdots + a_nb_n}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

And what we've shown earlier is:

Theorem: Let $\vec{a} = < a_1, a_2, \ldots, a_n >$ and $\vec{b} = < b_1, b_2, \ldots, b_n >$ be vectors in $\mathbb{R}^n$. And suppose θ is the smallest angle between them (note that, then $0 \le \theta \le \pi$). Then,

$$\boxed{\vec{a}\cdot\vec{b} \;=\; |\vec{a}||\vec{b}|\cos\theta}$$

e.g. $\vec{a} = \langle 1, 2\rangle$

$\vec{b} = \langle -3, 5\rangle$

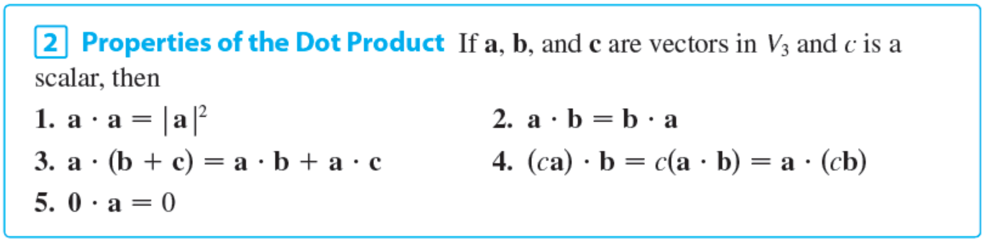
$\vec{a}\cdot\vec{b} = 1(-3) + 2(5) = \boxed{7}$

scalar!

Another important note: **THE DOT PRODUCT IS A SCALAR!!!!!!!**

In fact, sometimes the dot product is referred to as the "scalar product".
Also as the "inner product", but that's more of an abstract math thing.

Yet another thing to note (these work in $\mathbb{R}^n$ in general):



2 Properties of the Dot Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
4. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a} = 0$

We'll get to examples in a bit, but here's yet another thing to note:

Defn: $\vec{a} = < a_1, a_2, \ldots, a_n >$ and $\vec{b} = < b_1, b_2, \ldots, b_n >$ are said to be *orthogonal* iff the angle between them is $\frac{\pi}{2}$ radians. (We call this "perpendicular" in 2D).

This means that $\theta = \frac{\pi}{2}$ in our theorem, and so $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\frac{\pi}{2} = 0$. This leads to:

Defn/proposition/theorem: $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a} \cdot \vec{b} = 0$.

Let's take this time to state/re-state some important maxims

- $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a} \cdot \vec{b} = 0$
- $\vec{a}$ and $\vec{b}$ are paralel iff $\vec{a} = c\vec{b}$

Why the new name? Do we get so pretentious in calc 3 that we decide to rename everything? Well, sometimes, but in this case, a new name is warranted. Our idea of "perpendicular" is very lacking in dimensions higher than 2. We'll illustrate.

Now, on to examples:

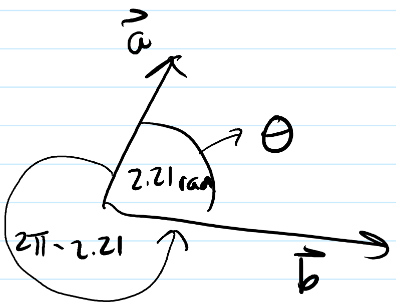
Example: Let $\vec{a} = < 1,2,3 >$ and $\vec{b} = < 1,0,-2 >$.
(a) Find $\vec{a} \cdot \vec{b}$
(b) Find the smallest angle θ between $\vec{a}$ and $\vec{b}$.

$$(a)\quad \vec{a} \cdot \vec{b} = \langle 1,2,3 \rangle \cdot \langle 1,0,-2 \rangle$$

Important!

$$= 1(1) + 2(0) + 3(-2)$$
$$= 1 + 0 - 6$$
$$= \boxed{-5}$$

$$(b)\quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{-5}{\sqrt{14}\sqrt{5}}$$
$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{-5}{\sqrt{14}\sqrt{5}}\right)}$$
$$\approx 2.21 \text{ rad.}$$



Example: Show that $< 1,3,5 >$ and $< -5,0,1 >$ are orthogonal.

$$\langle 1,3,5 \rangle \cdot \langle -5,0,1 \rangle = 1(-5) + 3(0) + 5(1)$$
$$= 0$$

$\Rightarrow$ the vectors are orthogonal!

Example: Find $\vec{a} \cdot \vec{b}$ if $|\vec{a}| = 2$, $|\vec{b}| = 3$, and $\theta = \frac{\pi}{4}$.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$\Rightarrow \vec{a} \cdot \vec{b} = 2(3) \cos \frac{\pi}{4}$$

$$= \boxed{3\sqrt{2}}$$

Example: Suppose $\vec{a} = < a_1, a_2 >$, where $a_1, a_2 \neq 0$. Find a non-zero vector $\vec{b}$ orthogonal to $\vec{a}$.

$c \neq 0$ a constant.

$$\vec{a} = \langle a_1, a_2 \rangle$$

$$\boxed{\vec{b} = c\langle -a_2, a_1 \rangle} \text{ or } \boxed{c\langle a_2, -a_1 \rangle}$$

then $\langle a_1, a_2 \rangle \cdot \langle -a_2, a_1 \rangle = -a_1 a_2 + a_1 a_2 = 0$

OR Set $\vec{b} = \langle b_1, b_2 \rangle$,

we want $\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = 0$

$$\Rightarrow a_1 b_1 + a_2 b_2 = 0$$

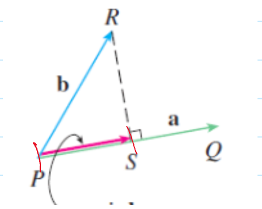
Vector and Scalar projections

Projections are very important and have many applications in physics, linear algebra, and geometry--and anything that depends on them, but we're just going to look at the math and not worry about the application.

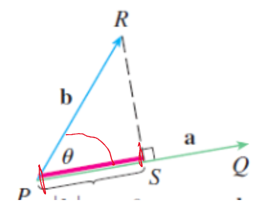
Sometimes, we have two vector and we might want to "project" perpendicularly, the shadow of one onto the other. The length of this shadow is called a **scalar projection** or **component**, while if we use another vector to run along this shadow, that vector is called **vector projection**.

The illustrations and formula boxes are taken from Stewart's Calculus:

Vector projections

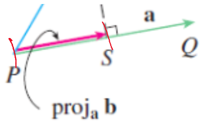


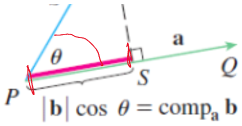
Scalar projections



length of shadow

$= |\vec{b}| \cos\theta$

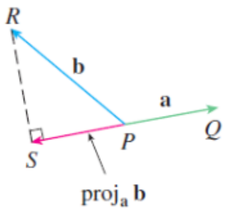




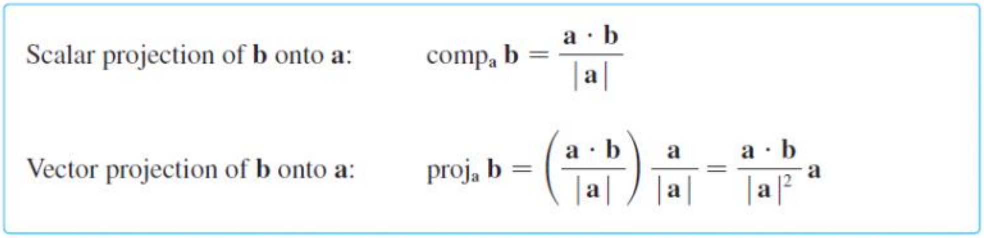
$$= |\vec{b}|\cos\theta$$

$$= \frac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{a}|}$$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$



What we call these and how they're denoted is described by:

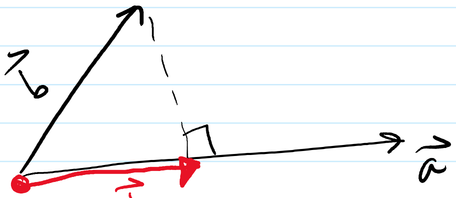


Scalar projection of **b** onto **a**: $\quad \text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\quad \text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}$

(I'll probably talk about where these formulas come from if I don't feel pressed for time. But a little bit of trig and knowledge of unit vectors will go a long way....)

Example: Find the vector and scalar projections of $\vec{b} = <1,-1,2>$ onto $\vec{a} = <3,0,1>$.



$\text{proj}_{\vec{a}}\vec{b}$ → It's length = $\text{comp}_{\vec{a}}\vec{b}$

Scalar projection: $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$$= \frac{<3,0,1>\cdot<1,-1,2>}{|<3,0,1>|}$$

$$= \frac{3+2}{\sqrt{10}}$$

$$= \frac{5}{\sqrt{10}}$$

Vector projection: $\vec{b} = \left(\vec{a}\cdot\vec{b}\right)\ \vec{a}$

Vector projection: $\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|}$

$$= \frac{5}{\sqrt{10}} \cdot \frac{\langle 3,0,1\rangle}{|\langle 3,0,1\rangle|}$$

$$= \frac{\langle 15,0,5\rangle}{10}$$

$$= \langle 1.5, 0, 0.5\rangle$$

$$= \langle \tfrac{3}{2}, 0, \tfrac{1}{2}\rangle$$