6. $\int_1^\infty \frac{1}{x^{\sqrt{2}}}\,dx$ is convergent.

7. If f is continuous, then $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$.

8. The Midpoint Rule is always more accurate than the Trapezoidal Rule.

9. (a) Every elementary function has an elementary derivative.
(b) Every elementary function has an elementary antiderivative.

10. If f is continuous on $[0, \infty)$ and $\int_1^\infty f(x)\,dx$ is convergent, then $\int_0^\infty f(x)\,dx$ is convergent.

11. If f is a continuous, decreasing function on $[1, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ is convergent.

12. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both convergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is convergent.

13. If $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ are both divergent, then $\int_a^\infty [f(x) + g(x)]\,dx$ is divergent.

14. If $f(x) \leqslant g(x)$ and $\int_0^\infty g(x)\,dx$ diverges, then $\int_0^\infty f(x)\,dx$ also diverges.

EXERCISES

1–40 ■ Evaluate the integral.

1. $\int_1^2 \frac{(x+1)^2}{x}\,dx$

2. $\int_1^2 \frac{x}{(x+1)^2}\,dx$

3. $\int_0^{\pi/2} \sin\theta\, e^{\cos\theta}\,d\theta$

4. $\int_0^{\pi/6} t\sin 2t\,dt$

5. $\int \frac{dt}{2t^2 + 3t + 1}$

6. $\int_1^2 x^5 \ln x\,dx$

7. $\int \frac{\sin(\ln t)}{t}\,dt$

8. $\int \frac{dx}{x^2\sqrt{1+x^2}}$

9. $\int_1^4 x^{3/2}\ln x\,dx$

10. $\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx$

11. $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$

12. $\int_{-\pi/4}^{\pi/4} \frac{\tan x}{4+x^2}\,dx$

13. $\int \frac{dx}{x^3 + x}$

14. $\int \frac{x^2+2}{x+2}\,dx$

15. $\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$

16. $\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$

17. $\int x\sec x\tan x\,dx$

18. $\int \frac{x^2+8x-3}{x^3+3x^2}\,dx$

19. $\int \frac{x+1}{9x^2+6x+5}\,dx$

20. $\int \frac{dt}{\sin^2 t + \cos 2t}$

21. $\int \frac{dx}{\sqrt{x^2-4x}}$

22. $\int \frac{x^3}{(x+1)^{10}}\,dx$

23. $\int \csc^4 4x\,dx$

24. $\int e^x\cos x\,dx$

25. $\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx$

26. $\int \tan^5\theta\sec^3\theta\,d\theta$

27. $\int_0^{\pi/2} \cos^3 x\sin 2x\,dx$

28. $\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx$

29. $\int_{-3}^{3} \frac{x}{1+|x|}\,dx$

30. $\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$

31. $\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx$

32. $\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx$

33. $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$

34. $\int (\arcsin x)^2\,dx$

35. $\int \frac{1}{\sqrt{x + x^{3/2}}}\,dx$

36. $\int \frac{1-\tan\theta}{1+\tan\theta}\,d\theta$

37. $\int (\cos x + \sin x)^2\cos 2x\,dx$

38. $\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

39. $\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx$

40. $\int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta$

41–50 ■ Evaluate the integral or show that it is divergent.

41. $\int_1^\infty \frac{1}{(2x+1)^3}\,dx$

42. $\int_1^\infty \frac{\ln x}{x^4}\,dx$

43. $\int_2^\infty \frac{dx}{x\ln x}$

44. $\int_2^6 \frac{y}{\sqrt{y-2}}\,dy$

45. $\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx$

46. $\int_0^1 \frac{1}{2-3x}\,dx$

47. $\int_0^1 \frac{x-1}{\sqrt{x}}\,dx$

48. $\int_{-1}^1 \frac{dx}{x^2-2x}$

49. $\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5}$

50. $\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$

51–54 ■ Use the Table of Integrals on the Reference Pages to evaluate the integral.

51. $\int \sqrt{4x^2-4x-3}\,dx$

52. $\int \csc^5 t\,dt$

Improper Integrals

Evaluate the improper integrals in Exercises 53–62.

53. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$ **54.** $\displaystyle\int_0^1 \ln x\,dx$

55. $\displaystyle\int_0^2 \frac{dy}{(y-1)^{2/3}}$ **56.** $\displaystyle\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}}$

57. $\displaystyle\int_3^\infty \frac{2\,du}{u^2-2u}$ **58.** $\displaystyle\int_1^\infty \frac{3v-1}{4v^3-v^2}\,dv$

59. $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$ **60.** $\displaystyle\int_{-\infty}^0 xe^{3x}\,dx$

61. $\displaystyle\int_{-\infty}^\infty \frac{dx}{4x^2+9}$ **62.** $\displaystyle\int_{-\infty}^\infty \frac{4\,dx}{x^2+16}$

Which of the improper integrals in Exercises 63–68 converge and which diverge?

63. $\displaystyle\int_6^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$ **64.** $\displaystyle\int_0^\infty e^{-u}\cos u\,du$

65. $\displaystyle\int_1^\infty \frac{\ln z}{z}\,dz$ **66.** $\displaystyle\int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$

67. $\displaystyle\int_{-\infty}^\infty \frac{2\,dx}{e^x+e^{-x}}$ **68.** $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)}$

Assorted Integrations

Evaluate the integrals in Exercises 69–116. The integrals are listed in random order so you need to decide which integration technique to use.

69. $\displaystyle\int \frac{x\,dx}{1+\sqrt{x}}$ **70.** $\displaystyle\int \frac{x^3+2}{4-x^2}\,dx$

71. $\displaystyle\int \sqrt{2x-x^2}\,dx$ **72.** $\displaystyle\int \frac{dx}{\sqrt{-2x-x^2}}$

73. $\displaystyle\int \frac{2-\cos x+\sin x}{\sin^2 x}\,dx$ **74.** $\displaystyle\int \sin^2\theta\cos^5\theta\,d\theta$

75. $\displaystyle\int \frac{9\,dv}{81-v^4}$ **76.** $\displaystyle\int_2^\infty \frac{dx}{(x-1)^2}$

77. $\displaystyle\int \theta\cos(2\theta+1)\,d\theta$ **78.** $\displaystyle\int \frac{x^3\,dx}{x^2-2x+1}$

79. $\displaystyle\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2}$ **80.** $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx$

81. $\displaystyle\int \frac{x\,dx}{\sqrt{2-x}}$ **82.** $\displaystyle\int \frac{\sqrt{1-v^2}}{v^2}\,dv$

83. $\displaystyle\int \frac{dy}{y^2-2y+2}$ **84.** $\displaystyle\int \frac{x\,dx}{\sqrt{8-2x^2-x^4}}$

85. $\displaystyle\int \frac{z+1}{z^2(z^2+4)}\,dz$ **86.** $\displaystyle\int x^2(x-1)^{1/3}\,dx$

87. $\displaystyle\int \frac{t\,dt}{\sqrt{9-4t^2}}$ **88.** $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

89. $\displaystyle\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$ **90.** $\displaystyle\int \tan^3 t\,dt$

91. $\displaystyle\int_1^\infty \frac{\ln y}{y^3}\,dy$ **92.** $\displaystyle\int y^{3/2}(\ln y)^2\,dy$

93. $\displaystyle\int e^{\ln\sqrt{x}}\,dx$ **94.** $\displaystyle\int e^\theta\sqrt{3+4e^\theta}\,d\theta$

95. $\displaystyle\int \frac{\sin 5t\,dt}{1+(\cos 5t)^2}$ **96.** $\displaystyle\int \frac{dv}{\sqrt{e^{2v}-1}}$

97. $\displaystyle\int \frac{dr}{1+\sqrt{r}}$ **98.** $\displaystyle\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx$

99. $\displaystyle\int \frac{x^3}{1+x^2}\,dx$ **100.** $\displaystyle\int \frac{x^2}{1+x^3}\,dx$

101. $\displaystyle\int \frac{1+x^2}{1+x^3}\,dx$ **102.** $\displaystyle\int \frac{1+x^2}{(1+x)^3}\,dx$

103. $\displaystyle\int \sqrt{x}\cdot\sqrt{1+\sqrt{x}}\,dx$ **104.** $\displaystyle\int \sqrt{1+\sqrt{1+x}}\,dx$

105. $\displaystyle\int \frac{1}{\sqrt{x}\cdot\sqrt{1+x}}\,dx$ **106.** $\displaystyle\int_0^{1/2} \sqrt{1+\sqrt{1-x^2}}\,dx$

107. $\displaystyle\int \frac{\ln x}{x+x\ln x}\,dx$ **108.** $\displaystyle\int \frac{1}{x\cdot\ln x\cdot\ln(\ln x)}\,dx$

109. $\displaystyle\int \frac{x^{\ln x}\ln x}{x}\,dx$ **110.** $\displaystyle\int (\ln x)^{\ln x}\left[\frac{1}{x}+\frac{\ln(\ln x)}{x}\right]dx$

111. $\displaystyle\int \frac{1}{x\sqrt{1-x^4}}\,dx$ **112.** $\displaystyle\int \frac{\sqrt{1-x}}{x}\,dx$

113. a. Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

b. Use part (a) to evaluate

$$\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x}\,dx.$$

114. $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx$ **115.** $\displaystyle\int \frac{\sin^2 x}{1+\sin^2 x}\,dx$

116. $\displaystyle\int \frac{1-\cos x}{1+\cos x}\,dx$

Chapter 8 Additional and Advanced Exercises

Evaluating Integrals

Evaluate the integrals in Exercises 1–6.

1. $\displaystyle\int (\sin^{-1}x)^2\,dx$

2. $\displaystyle\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

3. $\displaystyle\int x\sin^{-1}x\,dx$ **4.** $\displaystyle\int \sin^{-1}\sqrt{y}\,dy$

The functions that we have been dealing with in this book are called **elementary functions**. These are the polynomials, rational functions, power functions (x^n), exponential functions (b^x), logarithmic functions, trigonometric and inverse trigonometric functions, hyperbolic and inverse hyperbolic functions, and all functions that can be obtained from these by the five operations of addition, subtraction, multiplication, division, and composition. For instance, the function

$$f(x) = \sqrt{\frac{x^2 - 1}{x^3 + 2x - 1}} + \ln(\cosh x) - xe^{\sin 2x}$$

is an elementary function.

If f is an elementary function, then f' is an elementary function but $\int f(x)\,dx$ need not be an elementary function. Consider $f(x) = e^{x^2}$. Since f is continuous, its integral exists, and if we define the function F by

$$F(x) = \int_0^x e^{t^2}\,dt$$

then we know from Part 1 of the Fundamental Theorem of Calculus that

$$F'(x) = e^{x^2}$$

Thus $f(x) = e^{x^2}$ has an antiderivative F, but it has been proved that F is not an elementary function. This means that no matter how hard we try, we will never succeed in evaluating $\int e^{x^2}\,dx$ in terms of the functions we know. (In Chapter 11, however, we will see how to express $\int e^{x^2}\,dx$ as an infinite series.) The same can be said of the following integrals:

$$\int \frac{e^x}{x}\,dx \qquad \int \sin(x^2)\,dx \qquad \int \cos(e^x)\,dx$$

$$\int \sqrt{x^3 + 1}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

In fact, the majority of elementary functions don't have elementary antiderivatives. You may be assured, though, that the integrals in the following exercises are all elementary functions.

7.5 EXERCISES

1–82 Evaluate the integral.

1. $\displaystyle\int \frac{\cos x}{1 - \sin x}\,dx$

2. $\displaystyle\int_0^1 (3x + 1)^{\sqrt{2}}\,dx$

3. $\displaystyle\int_1^4 \sqrt{y}\,\ln y\,dy$

4. $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx$

5. $\displaystyle\int \frac{t}{t^4 + 2}\,dt$

6. $\displaystyle\int_0^1 \frac{x}{(2x + 1)^3}\,dx$

7. $\displaystyle\int_{-1}^1 \frac{e^{\arctan y}}{1 + y^2}\,dy$

8. $\displaystyle\int t \sin t \cos t\,dt$

9. $\displaystyle\int_2^4 \frac{x + 2}{x^2 + 3x - 4}\,dx$

10. $\displaystyle\int \frac{\cos(1/x)}{x^3}\,dx$

11. $\displaystyle\int \frac{1}{x^3\sqrt{x^2 - 1}}\,dx$

12. $\displaystyle\int \frac{2x - 3}{x^3 + 3x}\,dx$

13. $\displaystyle\int \sin^5 t\,\cos^4 t\,dt$

14. $\displaystyle\int \ln(1 + x^2)\,dx$

15. $\displaystyle\int x \sec x \tan x\,dx$

16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1 - x^2}}\,dx$

17. $\displaystyle\int_0^\pi t \cos^2 t\,dt$

18. $\displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$

19. $\displaystyle\int e^{x + e^x}\,dx$

20. $\displaystyle\int e^2\,dx$

21. $\displaystyle\int \arctan\sqrt{x}\,dx$

22. $\displaystyle\int \frac{\ln x}{x\sqrt{1 + (\ln x)^2}}\,dx$

23. $\int_0^1 (1 + \sqrt{x})^8\, dx$

24. $\int (1 + \tan x)^2 \sec x\, dx$

25. $\int_0^1 \frac{1 + 12t}{1 + 3t}\, dt$

26. $\int_0^1 \frac{3x^2 + 1}{x^3 + x^2 + x + 1}\, dx$

27. $\int \frac{dx}{1 + e^x}$

28. $\int \sin \sqrt{at}\, dt$

29. $\int \ln\left(x + \sqrt{x^2 - 1}\right) dx$

30. $\int_{-1}^2 |e^x - 1|\, dx$

31. $\int \sqrt{\frac{1 + x}{1 - x}}\, dx$

32. $\int_1^3 \frac{e^{3/x}}{x^2}\, dx$

33. $\int \sqrt{3 - 2x - x^2}\, dx$

34. $\int_{\pi/4}^{\pi/2} \frac{1 + 4 \cot x}{4 - \cot x}\, dx$

35. $\int_{-\pi/2}^{\pi/2} \frac{x}{1 + \cos^2 x}\, dx$

36. $\int \frac{1 + \sin x}{1 + \cos x}\, dx$

37. $\int_0^{\pi/4} \tan^3\theta \sec^2\theta\, d\theta$

38. $\int_{\pi/6}^{\pi/3} \frac{\sin\theta \cot\theta}{\sec\theta}\, d\theta$

39. $\int \frac{\sec\theta \tan\theta}{\sec^2\theta - \sec\theta}\, d\theta$

40. $\int_0^\pi \sin 6x \cos 3x\, dx$

41. $\int \theta \tan^2\theta\, d\theta$

42. $\int \frac{\tan^{-1} x}{x^2}\, dx$

43. $\int \frac{\sqrt{x}}{1 + x^3}\, dx$

44. $\int \sqrt{1 + e^x}\, dx$

45. $\int x^5 e^{-x^3} dx$

46. $\int \frac{(x - 1)e^x}{x^2}\, dx$

47. $\int x^3 (x - 1)^{-4}\, dx$

48. $\int_0^1 x\sqrt{2 - \sqrt{1 - x^2}}\, dx$

49. $\int \frac{1}{x\sqrt{4x + 1}}\, dx$

50. $\int \frac{1}{x^2\sqrt{4x + 1}}\, dx$

51. $\int \frac{1}{x\sqrt{4x^2 + 1}}\, dx$

52. $\int \frac{dx}{x(x^4 + 1)}$

53. $\int x^2 \sinh mx\, dx$

54. $\int (x + \sin x)^2\, dx$

55. $\int \frac{dx}{x + x\sqrt{x}}$

56. $\int \frac{dx}{\sqrt{x} + x\sqrt{x}}$

57. $\int x\sqrt[3]{x + c}\, dx$

58. $\int \frac{x \ln x}{\sqrt{x^2 - 1}}\, dx$

59. $\int \frac{dx}{x^4 - 16}$

60. $\int \frac{dx}{x^2\sqrt{4x^2 - 1}}$

61. $\int \frac{d\theta}{1 + \cos\theta}$

62. $\int \frac{d\theta}{1 + \cos^2\theta}$

63. $\int \sqrt{x}\, e^{\sqrt{x}}\, dx$

64. $\int \frac{1}{\sqrt{\sqrt{x} + 1}}\, dx$

65. $\int \frac{\sin 2x}{1 + \cos^4 x}\, dx$

66. $\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\sin x \cos x}\, dx$

67. $\int \frac{1}{\sqrt{x + 1} + \sqrt{x}}\, dx$

68. $\int \frac{x^2}{x^6 + 3x^3 + 2}\, dx$

69. $\int_1^{\sqrt{3}} \frac{\sqrt{1 + x^2}}{x^2}\, dx$

70. $\int \frac{1}{1 + 2e^x - e^{-x}}\, dx$

71. $\int \frac{e^{2x}}{1 + e^x}\, dx$

72. $\int \frac{\ln(x + 1)}{x^2}\, dx$

73. $\int \frac{x + \arcsin x}{\sqrt{1 - x^2}}\, dx$

74. $\int \frac{4^x + 10^x}{2^x}\, dx$

75. $\int \frac{dx}{x \ln x - x}$

76. $\int \frac{x^2}{\sqrt{x^2 + 1}}\, dx$

77. $\int \frac{xe^x}{\sqrt{1 + e^x}}\, dx$

78. $\int \frac{1 + \sin x}{1 - \sin x}\, dx$

79. $\int x \sin^2 x \cos x\, dx$

80. $\int \frac{\sec x \cos 2x}{\sin x + \sec x}\, dx$

81. $\int \sqrt{1 - \sin x}\, dx$

82. $\int \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$

83. The functions $y = e^{x^2}$ and $y = x^2 e^{x^2}$ don't have elementary antiderivatives, but $y = (2x^2 + 1)e^{x^2}$ does. Evaluate $\int (2x^2 + 1)e^{x^2} dx$.

84. We know that $F(x) = \int_0^x e^{e^t} dt$ is a continuous function by FTC1, though it is not an elementary function. The functions

$$\int \frac{e^x}{x}\, dx \qquad \text{and} \qquad \int \frac{1}{\ln x}\, dx$$

are not elementary either, but they can be expressed in terms of F. Evaluate the following integrals in terms of F.

(a) $\int_1^2 \frac{e^x}{x}\, dx$ (b) $\int_2^3 \frac{1}{\ln x}\, dx$

7.6 Integration Using Tables and Computer Algebra Systems

In this section we describe how to use tables and computer algebra systems to integrate functions that have elementary antiderivatives. You should bear in mind, though, that even

7 REVIEW

CONCEPT CHECK

Answers to the Concept Check can be found on the back endpapers.

1. State the rule for integration by parts. In practice, how do you use it?

2. How do you evaluate $\int \sin^m x \cos^n x\, dx$ if m is odd? What if n is odd? What if m and n are both even?

3. If the expression $\sqrt{a^2 - x^2}$ occurs in an integral, what substitution might you try? What if $\sqrt{a^2 + x^2}$ occurs? What if $\sqrt{x^2 - a^2}$ occurs?

4. What is the form of the partial fraction decomposition of a rational function $P(x)/Q(x)$ if the degree of P is less than the degree of Q and $Q(x)$ has only distinct linear factors? What if a linear factor is repeated? What if $Q(x)$ has an irreducible quadratic factor (not repeated)? What if the quadratic factor is repeated?

5. State the rules for approximating the definite integral $\int_a^b f(x)\, dx$ with the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule. Which would you expect to give the best estimate? How do you approximate the error for each rule?

6. Define the following improper integrals.
   (a) $\int_a^\infty f(x)\, dx$ (b) $\int_{-\infty}^b f(x)\, dx$ (c) $\int_{-\infty}^\infty f(x)\, dx$

7. Define the improper integral $\int_a^b f(x)\, dx$ for each of the following cases.
   (a) f has an infinite discontinuity at a.
   (b) f has an infinite discontinuity at b.
   (c) f has an infinite discontinuity at c, where $a < c < b$.

8. State the Comparison Theorem for improper integrals.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\dfrac{x(x^2+4)}{x^2-4}$ can be put in the form $\dfrac{A}{x+2} + \dfrac{B}{x-2}$.

2. $\dfrac{x^2+4}{x(x^2-4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-2}$.

3. $\dfrac{x^2+4}{x^2(x-4)}$ can be put in the form $\dfrac{A}{x^2} + \dfrac{B}{x-4}$.

4. $\dfrac{x^2-4}{x(x^2+4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2+4}$.

5. $\displaystyle\int_0^4 \frac{x}{x^2-1}\, dx = \tfrac{1}{2} \ln 15$

6. $\displaystyle\int_1^\infty \frac{1}{x^{\sqrt{2}}}\, dx$ is convergent.

7. If f is continuous, then $\int_{-\infty}^\infty f(x)\, dx = \lim_{t\to\infty} \int_{-t}^t f(x)\, dx$.

8. The Midpoint Rule is always more accurate than the Trapezoidal Rule.

9. (a) Every elementary function has an elementary derivative.
   (b) Every elementary function has an elementary antiderivative.

10. If f is continuous on $[0, \infty)$ and $\int_1^\infty f(x)\, dx$ is convergent, then $\int_0^\infty f(x)\, dx$ is convergent.

11. If f is a continuous, decreasing function on $[1, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\, dx$ is convergent.

12. If $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ are both convergent, then $\int_a^\infty [f(x) + g(x)]\, dx$ is convergent.

13. If $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ are both divergent, then $\int_a^\infty [f(x) + g(x)]\, dx$ is divergent.

14. If $f(x) \leqslant g(x)$ and $\int_0^\infty g(x)\, dx$ diverges, then $\int_0^\infty f(x)\, dx$ also diverges.

EXERCISES

Note: Additional practice in techniques of integration is provided in Exercises 7.5.

1–40 Evaluate the integral.

1. $\displaystyle\int_1^2 \frac{(x+1)^2}{x}\, dx$

2. $\displaystyle\int_1^2 \frac{x}{(x+1)^2}\, dx$

3. $\displaystyle\int \frac{e^{\sin x}}{\sec x}\, dx$

4. $\displaystyle\int_0^{\pi/6} t \sin 2t\, dt$

5. $\displaystyle\int \frac{dt}{2t^2 + 3t + 1}$

6. $\displaystyle\int_1^2 x^5 \ln x\, dx$

7. $\displaystyle\int_0^{\pi/2} \sin^3\theta \cos^2\theta\, d\theta$

8. $\displaystyle\int \frac{dx}{\sqrt{e^x - 1}}$

9. $\displaystyle\int \frac{\sin(\ln t)}{t}\,dt$

10. $\displaystyle\int_0^1 \frac{\sqrt{\arctan x}}{1 + x^2}\,dx$

11. $\displaystyle\int_1^2 \frac{\sqrt{x^2 - 1}}{x}\,dx$

12. $\displaystyle\int \frac{e^{2x}}{1 + e^{4x}}\,dx$

13. $\displaystyle\int e^{\sqrt[3]{x}}\,dx$

14. $\displaystyle\int \frac{x^2 + 2}{x + 2}\,dx$

15. $\displaystyle\int \frac{x - 1}{x^2 + 2x}\,dx$

16. $\displaystyle\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta$

17. $\displaystyle\int x\cosh x\,dx$

18. $\displaystyle\int \frac{x^2 + 8x - 3}{x^3 + 3x^2}\,dx$

19. $\displaystyle\int \frac{x + 1}{9x^2 + 6x + 5}\,dx$

20. $\displaystyle\int \tan^5\theta\,\sec^3\theta\,d\theta$

21. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4x}}$

22. $\displaystyle\int \cos\sqrt{t}\,dt$

23. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + 1}}$

24. $\displaystyle\int e^x\cos x\,dx$

25. $\displaystyle\int \frac{3x^3 - x^2 + 6x - 4}{(x^2 + 1)(x^2 + 2)}\,dx$

26. $\displaystyle\int x\sin x\cos x\,dx$

27. $\displaystyle\int_0^{\pi/2} \cos^3 x\,\sin 2x\,dx$

28. $\displaystyle\int \frac{\sqrt[3]{x} + 1}{\sqrt[3]{x} - 1}\,dx$

29. $\displaystyle\int_{-3}^{3} \frac{x}{1 + |x|}\,dx$

30. $\displaystyle\int \frac{dx}{e^x\sqrt{1 - e^{-2x}}}$

31. $\displaystyle\int_0^{\ln 10} \frac{e^x\sqrt{e^x - 1}}{e^x + 8}\,dx$

32. $\displaystyle\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx$

33. $\displaystyle\int \frac{x^2}{(4 - x^2)^{3/2}}\,dx$

34. $\displaystyle\int (\arcsin x)^2\,dx$

35. $\displaystyle\int \frac{1}{\sqrt{x + x^{3/2}}}\,dx$

36. $\displaystyle\int \frac{1 - \tan\theta}{1 + \tan\theta}\,d\theta$

37. $\displaystyle\int (\cos x + \sin x)^2\cos 2x\,dx$

38. $\displaystyle\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx$

39. $\displaystyle\int_0^{1/2} \frac{xe^{2x}}{(1 + 2x)^2}\,dx$

40. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta$

41–50 Evaluate the integral or show that it is divergent.

41. $\displaystyle\int_1^{\infty} \frac{1}{(2x + 1)^3}\,dx$

42. $\displaystyle\int_1^{\infty} \frac{\ln x}{x^4}\,dx$

43. $\displaystyle\int_2^{\infty} \frac{dx}{x\ln x}$

44. $\displaystyle\int_2^6 \frac{y}{\sqrt{y - 2}}\,dy$

45. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx$

46. $\displaystyle\int_0^1 \frac{1}{2 - 3x}\,dx$

47. $\displaystyle\int_0^1 \frac{x - 1}{\sqrt{x}}\,dx$

48. $\displaystyle\int_{-1}^1 \frac{dx}{x^2 - 2x}$

49. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4x^2 + 4x + 5}$

50. $\displaystyle\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx$

51–52 Evaluate the indefinite integral. Illustrate and check that your answer is reasonable by graphing both the function and its antiderivative (take $C = 0$).

51. $\displaystyle\int \ln(x^2 + 2x + 2)\,dx$

52. $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 1}}\,dx$

53. Graph the function $f(x) = \cos^2 x\,\sin^3 x$ and use the graph to guess the value of the integral $\int_0^{2\pi} f(x)\,dx$. Then evaluate the integral to confirm your guess.

CAS **54.** (a) How would you evaluate $\int x^5 e^{-2x}\,dx$ by hand? (Don't actually carry out the integration.)
(b) How would you evaluate $\int x^5 e^{-2x}\,dx$ using tables? (Don't actually do it.)
(c) Use a CAS to evaluate $\int x^5 e^{-2x}\,dx$.
(d) Graph the integrand and the indefinite integral on the same screen.

55–58 Use the Table of Integrals on the Reference Pages to evaluate the integral.

55. $\displaystyle\int \sqrt{4x^2 - 4x - 3}\,dx$

56. $\displaystyle\int \csc^5 t\,dt$

57. $\displaystyle\int \cos x\sqrt{4 + \sin^2 x}\,dx$

58. $\displaystyle\int \frac{\cot x}{\sqrt{1 + 2\sin x}}\,dx$

59. Verify Formula 33 in the Table of Integrals (a) by differentiation and (b) by using a trigonometric substitution.

60. Verify Formula 62 in the Table of Integrals.

61. Is it possible to find a number n such that $\int_0^{\infty} x^n\,dx$ is convergent?

62. For what values of a is $\int_0^{\infty} e^{ax}\cos x\,dx$ convergent? Evaluate the integral for those values of a.

63–64 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson's Rule with $n = 10$ to approximate the given integral. Round your answers to six decimal places.

63. $\displaystyle\int_2^4 \frac{1}{\ln x}\,dx$

64. $\displaystyle\int_1^4 \sqrt{x}\cos x\,dx$

65. Estimate the errors involved in Exercise 63, parts (a) and (b). How large should n be in each case to guarantee an error of less than 0.00001?

66. Use Simpson's Rule with $n = 6$ to estimate the area under the curve $y = e^x/x$ from $x = 1$ to $x = 4$.

researchers look at a variety of graphs to try to distinguish randomness from deterministic chaos. For example, iterate the function $f(x) = 4x(1 - x)$ starting at $x = 0.1$. That is, compute $f(0.1) = 0.36$, $f(0.36) = 0.9216$, $f(0.9216) \approx 0.289$ and so on. Iterate 50 times and record how many times each first digit occurs. (So far, we've got a 1, a 3, a 9 and a 2.) If the process were truly random, the digits would occur about the same number of times. Does this seem to be happening? To unmask this process as nonrandom, you can draw a **phase portrait.** To do this, take consecutive iterates as coordinates of a point (x, y) and plot the points. The first three points are (0.1, 0.36), (0.36, 0.9216) and (0.9216, 0.289). Describe the (nonrandom) pattern that appears, identifying it as precisely as possible.

2. Suppose that a spring is oscillating up and down with vertical position given by $u(t) = \sin t$. If you pick a random time and look at the position of the spring, would you be more likely to find the spring near an extreme ($u = 1$ or $u = -1$) or near the middle ($u = 0$)? The pdf is inversely proportional to speed. (Why is this reasonable?) Show that speed is given by $|\cos t| = \sqrt{1 - u^2}$, so the pdf is $f(u) = c/\sqrt{1 - u^2}$, $-1 \le u \le 1$, for some constant c. Show that $c = 1/\pi$, then graph $f(x)$ and describe the positions in which the spring is likely to be found. Use this result to explain the following. If you are driving in a residential neighborhood, you are more likely to meet a car coming the other way at an intersection than in the middle of a block.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Integration by parts	Reduction formula
Partial fractions decomposition	CAS
Improper integral	Integral converges
Integral diverges	Comparison Test
	Probability density function

TRUE OR FALSE

State whether each statement is true or false, and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to a new statement that is true.

1. Integration by parts works only for integrals of the form $\int f(x)g(x)\,dx$.
2. For an integral of the form $\int x f(x)\,dx$, always use integration by parts with $u = x$.
3. The trigonometric techniques in section 7.3 are all versions of substitution.
4. If an integrand contains a factor of $\sqrt{1 - x^2}$, you should substitute $x = \sin\theta$.
5. If p and q are polynomials, then any integral of the form $\displaystyle\int \frac{p(x)}{q(x)}\,dx$ can be evaluated.
6. With an extensive integral table, you don't need to know any integration techniques.
7. If $f(x)$ has a vertical asymptote at $x = a$, then $\int_a^b f(x)\,dx$ diverges for any b.
8. If $\lim_{x\to\infty} f(x) = L \ne 0$, then $\int_1^\infty f(x)\,dx$ diverges.
9. The mean of a random variable is always larger than the median.
10. L'Hôpital's Rule states that the limit of the derivative equals the limit of the function.

In exercises 1–44, evaluate the integral.

1. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

2. $\displaystyle\int \frac{\sin(1/x)}{x^2}\,dx$

3. $\displaystyle\int \frac{x^2}{\sqrt{1 - x^2}}\,dx$

4. $\displaystyle\int \frac{2}{\sqrt{9 - x^2}}\,dx$

5. $\displaystyle\int x^2 e^{-3x}\,dx$

6. $\displaystyle\int x^2 e^{-x^3}\,dx$

7. $\displaystyle\int \frac{x}{1 + x^4}\,dx$

8. $\displaystyle\int \frac{x^3}{1 + x^4}\,dx$

9. $\displaystyle\int \frac{x^3}{4 + x^4}\,dx$

10. $\displaystyle\int \frac{x}{4 + x^4}\,dx$

11. $\displaystyle\int e^{2\ln x}\,dx$

12. $\displaystyle\int \cos 4x\,dx$

13. $\displaystyle\int_0^1 x \sin 3x\,dx$

14. $\displaystyle\int_0^1 x \sin 4x^2\,dx$

Review Exercises

15. $\int_0^{\pi/2} \sin^4 x\,dx$

16. $\int_0^{\pi/2} \cos^3 x\,dx$

17. $\int_{-1}^{1} x \sin \pi x\,dx$

18. $\int_0^1 x^2 \cos \pi x\,dx$

19. $\int_1^2 x^3 \ln x\,dx$

20. $\int_0^{\pi/4} \sin x \cos x\,dx$

21. $\int \cos x \sin^2 x\,dx$

22. $\int \cos x \sin^3 x\,dx$

23. $\int \cos^3 x \sin^3 x\,dx$

24. $\int \cos^4 x \sin^3 x\,dx$

25. $\int \tan^2 x \sec^4 x\,dx$

26. $\int \tan^3 x \sec^2 x\,dx$

27. $\int \sqrt{\sin x} \cos^3 x\,dx$

28. $\int \tan^3 x \sec^3 x\,dx$

29. $\int \frac{2}{8+4x+x^2}\,dx$

30. $\int \frac{3}{\sqrt{-2x-x^2}}\,dx$

31. $\int \frac{2}{x^2\sqrt{4-x^2}}\,dx$

32. $\int \frac{x}{\sqrt{9-x^2}}\,dx$

33. $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$

34. $\int \frac{x^3}{\sqrt{x^2-9}}\,dx$

35. $\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

36. $\int \frac{4}{\sqrt{x+9}}\,dx$

37. $\int \frac{x+4}{x^2+3x+2}\,dx$

38. $\int \frac{5x+6}{x^2+x-12}\,dx$

39. $\int \frac{4x^2+6x-12}{x^3-4x}\,dx$

40. $\int \frac{5x^2+2}{x^3+x}\,dx$

41. $\int e^x \cos 2x\,dx$

42. $\int x^3 \sin x^2\,dx$

43. $\int x\sqrt{x^2+1}\,dx$

44. $\int \sqrt{1-x^2}\,dx$

In exercises 45–50, find the partial fractions decomposition.

45. $\frac{4}{x^2-3x-4}$

46. $\frac{2x}{x^2+x-6}$

47. $\frac{-6}{x^3+x^2-2x}$

48. $\frac{x^2-2x-2}{x^3+x}$

49. $\frac{x-2}{x^2+4x+4}$

50. $\frac{x^2-2}{(x^2+1)^2}$

In exercises 51–60, use the Table of Integrals to find the integral.

51. $\int e^{3x}\sqrt{4+e^{2x}}\,dx$

52. $\int x\sqrt{x^4-4}\,dx$

53. $\int \sec^4 x\,dx$

54. $\int \tan^5 x\,dx$

55. $\int \frac{4}{x(3-x)^2}\,dx$

56. $\int \frac{\cos x}{\sin^2 x(3+4\sin x)}\,dx$

57. $\int \frac{\sqrt{9+4x^2}}{x^2}\,dx$

58. $\int \frac{x^2}{\sqrt{4-9x^2}}\,dx$

59. $\int \frac{\sqrt{4-x^2}}{x}\,dx$

60. $\int \frac{x^2}{(x^6-4)^{3/2}}\,dx$

In exercises 61–68, determine whether the integral converges or diverges. If it converges, find the limit.

61. $\int_0^1 \frac{x}{x^2-1}\,dx$

62. $\int_4^{10} \frac{2}{\sqrt{x-4}}\,dx$

63. $\int_1^{\infty} \frac{3}{x^2}\,dx$

64. $\int_1^{\infty} xe^{-3x}\,dx$

65. $\int_0^{\infty} \frac{4}{4+x^2}\,dx$

66. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$

67. $\int_{-2}^{2} \frac{3}{x^2}\,dx$

68. $\int_{-2}^{2} \frac{x}{1-x^2}\,dx$

In exercises 69–76, find the limit.

69. $\lim_{x\to 1} \frac{x^3-1}{x^2-1}$

70. $\lim_{x\to 0} \frac{\sin x}{x^2+3x}$

71. $\lim_{x\to\infty} \frac{e^{2x}}{x^4+2}$

72. $\lim_{x\to\infty} (x^2e^{-3x})$

73. $\lim_{x\to 2^+} \left|\frac{x+1}{x-2}\right|^{\sqrt{x^2-4}}$

74. $\lim_{x\to\infty} x\ln(1+1/x)$

75. $\lim_{x\to 0^+} (\tan x \ln x)$

76. $\lim_{x\to 0} \frac{\tan^{-1} x}{\sin^{-1} x}$

77. Show that $f(x) = x + 2x^3$ is a pdf on the interval [0, 1].

78. Show that $f(x) = \frac{8}{3}e^{-2x}$ is a pdf on the interval [0, ln 2].

79. Find the value of c such that $f(x) = \frac{c}{x^2}$ is a pdf on the interval [1, 2].

80. Find the value of c such that $f(x) = ce^{-2x}$ is a pdf on the interval [0, 4].

81. The lifetime of a lightbulb has pdf $f(x) = 4e^{-4x}$ (x in years). Find the probability that the lightbulb lasts (a) less than 6 months; (b) between 6 months and 1 year.

82. The lifetime of an organism has pdf $f(x) = 9xe^{-3x}$ (x in years). Find the probability that the organism lasts (a) less than 2 months; (b) between 3 months and 1 year.

83. Find the (a) mean and (b) median of a random variable with pdf $f(x) = x + 2x^3$ on the interval [0, 1].