Math 346 - Final Exam - Fall, 2022

Part 1 - Answer all parts of the following five questions.

1. (10 points) What is the rank of the following matrix?

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Does this matrix have an inverse? Explain your answer.

2. (12 points) In the vector space $\mathcal{P}_3$ of polynomials of degree at most three show that $\{t^3 + 2t^2 - t + 1, 4t^2, t^3 + 2t^2 + 1\}$ is a linearly independent set. Does it form a basis for $\mathcal{P}_3$ (Justify your answer)?

3. (12 points) (a) For the homogeneous system with the following coefficient matrix, obtain the general solution and use it to get a basis for the solution space.

$$\begin{pmatrix} 3 & -6 & 1 & 3 & 0 & 3 \\ 0 & 0 & 1 & 0 & -3 & 0 \\ 1 & -2 & 0 & 1 & 1 & 1 \end{pmatrix}$$

(b) For the matrix of part (a) obtain a basis for the column space and a basis for the row space.

4. (12 points) For the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$,

(a) Compute the inverse and the determinant of A.
(b) <u>Use the inverse</u> to solve the system:

$$\begin{aligned} 3x + y + z &= 2, \\ 2x + y &= -1, \\ x + y + z &= 0. \end{aligned}$$

5. (12 points) For the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ -1 & -3 & 2 \\ 2 & -2 & 2 \end{pmatrix}$$

Compute the eigenvalues and a basis for each eigenspace.

Compute an invertible matrix Q and a diagonal matrix D such that $AQ = QD$, or, equivalently $Q^{-1}AQ = D$ (You need not compute Q^{-1}).

6. (12 points) For the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

use the Gram-Schmidt procedure to obtain an orthonormal basis for the column space and use it to compute the orthogonal projection of the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto the column space.

Part 2 - Answer all parts of three of the following five questions (10 points each).

6. (a) Define what it means for the list of vectors $\{v_1, v_2, v_3, v_4\}$ to be linearly independent.

(b) If v_1 is a linear combination of $\{v_2, v_3, v_4\}$, show that the list $\{v_1, v_2, v_3, v_4\}$ is linearly dependent, i.e. not linearly independent.

7. (a) For the row operation: to Row 2 add 4 times a copy of Row 3, write down the associated 4×4 elementary matrix <u>and</u> its inverse.

(b) Let A and B be invertible $n \times n$ matrices. Show that the product AB is invertible.

(c) Suppose that A and B are 5×5 matrices with $det(A) = 3, det(B) = 5$. Compute $det(2A)$ and $det(B^T A^2)$.

8. (a) Define what it means for an $n \times n$ matrix to be (i) orthogonal, (ii) symmetric.

(b) Show that if X_1 and X_2 are eigenvectors of a symmetric matrix A with different eigenvalues, then the pair $\{X_1, X_2\}$ is linearly independent.

9. Let $L : V \to W$ be a map between vector spaces.

(a) Define what it means for L to be *linear.*

(b) For the linear map $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$L(x_1, x_2, x_3) = (2x_1 + x_2, x_3 - x_2),$$

compute the matrix associated with the linear map, using the basis $B_3 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$ for $\mathbb{R}^3$ and the basis $B_2 = \{(1, 0), (0, 1)\}$ for $\mathbb{R}^2$.

10. Find the least squares approximating line $y = z_0 + z_1 x$ for the set of data points $(x, y) = (-1, 0), (0, 0), (1, 1)$.