EUCLID'S GEOMETRY

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Euclid first codified the procedures and results of geometry, and he did such a good job that even today it is hard to improve on his presentation. He lived around 300 BC between the time of Plato (died 347 BC) and Archimedes (287-212 BC).

1 Undefined Notions in Plane Geometry

1. Points
2. Lines, rays, and line segments
3. Congruence of plane figures

The undefined notion of congruence can be defined if we have a definition of a rigid motion. Then two geometric figures are congruent if there is a rigid motion which takes one figure into another.

2 Euclidean Axioms

1. Two points determine one and only one line.
2. Given a point on a line, there is a line segment of any given length starting at the given point and contained in the line.
3. There is one and only one circle with a given center and a given radius.
4. Any two right angles have the same measure. (An angle is a right angle if, when one side is extended, the resulting angles are congruent.)
5. (Parallel postulate) Given a point and a line not containing the point, there is one and only one line through the point parallel to the given line.

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Early mathematicians drew a distinction between an *axiom* and a *postulate*. Basically a postulate asserted that an object could be constructed while an axiom asserted a fact about an object. I have not drawn that distinction primarily because their view of geometry as an actual physically constructible set of objects and facts about these objects glosses over the issue of exactly what the results of Euclidean geometry represent. It took much time and effort by many mathematicians to reformulate this material in a truly satisfactory logical form. For instance, the common elements given above suggest some of these difficulties. Heath states 4) as "Things which coincide with one another are equal to one another." But what exactly does it mean for two things to "coincide"? Apparently, according to Euclid, this meant that one thing could be superimposed upon another and, if line segments and points matched, then the things were equal. The usual meaning of "superimposed" is that there is a motion of the plane which does not change lengths or angles, that is, a *rigid* motion of the plane, which places one thing upon the other. Since there are other classes of motions of the plane, e.g.using polar coordinates and changing the scaling along the radial axis, which may be used to make objects coincide, there must be substantial further discussion of *coincide* and *equal* before we can be truly satisfied with Euclidean geometry.

There are two websites that contain marvelous material on Euclid's *Elements*. They both have Java applets that carry out the construction of diagrams used in the proofs of the various results. You should look at them both. The first one is David Joyce's website located at Clark University. It can be reached through the second one which is at the University of British Columbia. This address is: http://www.math.ubc.ca/people/faculty/cass/Euclid/

Book I

Proposition 1 *An equilateral triangle can be constructed with a given line segment as base.*

Proposition 2 *Given a line segment and a point, a congruent line segment can be constructed from the given point.*

Proposition 3 *Given a line segment and a ray, a line segment congruent to the given one can be cut from the ray.*

Proposition 4 *(SAS) If two triangles have two sides and the included angle of one congruent to two sides and the included angle of the other, then the triangles are congruent.*

Proposition 5 *The base angles of an isosceles triangle are congruent as are the exterior angles underneath the base if the congruent sides are extended.*

Proposition 6 *If two angles of a triangle are congruent, then the sides opposite these angles are congruent.*

Proposition 7 *(Uniqueness of $\triangle$) At most one triangle can be constructed with a given line segment as base by using two line segments of given lengths starting from given endpoints of the base.*

Proposition 8 *(SSS). If two triangles have all three sides of one congruent to all three sides of the other, the triangles are congruent.*

Proposition 9 *A given angle can be bisected.*

Proposition 10 *A given line segment can be bisected.*

Proposition 11 *A perpendicular to a given line at a given point on the line can be constructed.*

Proposition 12 *Given a line and a point not on the line, a perpendicular to the given line passing through the given point can be constructed.*

Proposition 13 *The two angles formed by a ray starting at a point on a given line are either both right angles or add up to $180°$.*

Proposition 14 *Let two rays start from a given point on a line and lie on opposite sides of the line. If the sum of the adjacent angles formed by the two rays and the line is $180°$, then the rays themselves lie on a straight line.*

Proposition 15 *If two lines cross each other, they make equal opposite angles.*

Proposition 16 *If one side of a triangle is extended, then the exterior angle at that vertex of the triangle is greater than either of the two opposite interior angles of the triangle.*

Proposition 17 *The sum of the measures of any two angles in a triangle is less than $180°$.*

Proposition 18 *A longer side of a triangle is opposite a greater angle of the triangle.*

Proposition 19 *A greater angle of a triangle is opposite a longer side of the triangle.*

Proposition 20 *The sum of the lengths of any two sides of a triangle is greater than the length of the third side.*

Proposition 21 *If two triangles are constructed on the same base and one is contained entirely within the other, then the sum of the lengths of the two sides of the interior triangle is less than the sum of the lengths of the two sides of the exterior triangle and the interior angle they make is greater than the corresponding angle in the exterior triangle.*

Proposition 22 *Given line segments such that the length of any two of them is greater than the length of the third, a triangle with these segments as sides can be constructed.*

Proposition 23 *Given an angle, a congruent angle can be constructed with a given line and point on the line as one side and vertex.*

Proposition 24 *If two triangles have two sides on one congruent to two sides of the other but the angle determined by the two sides in the first triangle is smaller in measure that the corresponding angle in the second triangle, then the side opposite the angle in the first triangle is shorter than the corresponding side of the second triangle.*

Proposition 25 *If two triangles have two sides of one congruent to two sides of the other but the base of the first is shorter than the base of the second triangle, then the angle the sides determine in the first triangle is smaller than the corresponding angle in the second.*

Proposition 26 *(ASA). If two triangles have one side and two angles of one triangle congruent to one side and the corresponding two angles of the other, then the triangles are congruent.*

Proposition 27 *If a transversal to two lines makes equal alternate interior angles, then the two lines are parallel.*

Proposition 28 *If a transversal to two lines makes the exterior angle of one equal to the interior angle of the other on the same side of the transversal or the sum of the measures of the two interior angles on the same side of the transversal is 180°, then the lines are parallel.*

Proposition 29 *A transversal to two parallel lines makes alternate interior angles equal, the exterior angle of one equal to the interior angle of the other on the same side of the transversal, and the sum of the measures of the two interior angles on the same side of the transversal equal to 180°.*

Proposition 30 *Two lines parallel to a third are parallel to each other.*

Proposition 31 *A line parallel to a given line can be constructed through a given point not on the given line.*

Proposition 32 *If one side of a triangle is extended, the exterior angle is equal to the sum of the two opposite interior angles. Thus the sum of the angles in a triangle is 180°.*

Proposition 33 *The line segments joining respective endpoints of parallel, congruent line segments are parallel and congruent.*

Proposition 34 *Opposite sides and angles of a parallelogram are congruent, and a diagonal divides a parallelogram into two triangles of equal area.*

Proposition 35 *Parallelograms with the same base which lie between the same parallels have equal areas.*

Proposition 36 *Parallelograms with congruent bases which lie between the same parallels have equal area.*

Proposition 37 *Triangles with the same base which lie between the same parallels have equal areas.*

Proposition 38 *Triangles with congruent bases which lie between the same parallels have equal area.*

Proposition 39 *Triangles of equal area with the same base and on the same side of the base lie on the same parallels.*

Proposition 40 *Triangles of equal area and congruent bases on the same side of a line lie on the same parallels.*

Proposition 41 *If a parallelogram has the same base as a triangle and they lie on the same parallels, then the area of the parallelogram is twice the area of the triangle.*

Proposition 42 *A parallelogram can be constructed with a given angle at one vertex whose area is the same as the area of a given triangle.*

Proposition 43 *If a parallelogram is divided into four parallelograms meeting at a point on a diagonal, the areas of the two parallelograms which do not contain a segment of the diagonal are equal.*

Proposition 44 *A parallelogram can be constructed with a given base and a given angle on the base whose area is the same as that of a given triangle.*

Proposition 45 *A parallelogram can be constructed with a given angle at one vertex whose area is the same as the area of any rectilineal figure.*

Proposition 46 *A square can be constructed with a given line segment as base.*

Proposition 47 *(Pythagorean Theorem). The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides.*

Proposition 48 *If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the angle defined by the two sides is a right angle.*

Definition 49 *A circle is the set of points equidistant from a given point (the center of the circle).*

Definition 50 *A chord of a circle is the line segment joining two distinct points of the circle.*

Definition 51 *A secant line or ray is a line or ray that contains two distinct points of a circle.*

Definition 52 *A tangent line or ray to a given circle is a line or ray which touches the given circle at only one point. (Here the ray must be extended to a line before applying the test.)*

Definition 53 *A central angle is an angle with vertex at the center of a circle. Such an angle determines a corresponding arc on the circle.*

Definition 54 *If A and B are two points on a circle that are not endpoints of a diameter, the minor arc $\widehat{AB}$ consists of A and B and the points on the smaller arc of the circle between A and B while the major arc $\widehat{AB}$ consists of A and B and the points on the larger arc of the circle between A and B. An arc $\widehat{ABC}$ where A, B, and C are points on a circle denotes the are with endpoints A and C containing B. The measure of an arc $\widehat{ABC}$ is the measure of the corresponding central angle $\angle AOC$.*

Definition 55 *The angle $\angle LMN$ is said to be the inscribed angle on an arc $\widehat{ABC}$ if M lies on the arc $\widehat{ABC}$ but is not an endpoint and the sides of the angle, LM and MN, pass through the points A and C. Such an angle is said to lie upon the other arc. (See Proposition 20.)*

Remark 56 *Note that the distance from a point P to a line (or line segment) $\overleftrightarrow{AB}$ is measured along the line through P perpendicular to $\overleftrightarrow{AB}$.*

Proposition 57 *The center of a circle can be constructed.*

Proposition 58 *If A and B are two distinct points on a circle, then the chord $\overline{AB}$ lies entirely inside the circle.*

Proposition 59 *In a circle, a diameter bisects a chord if and only if it is perpendicular to the chord.*

Proposition 60 *Two chords which are not diameters cannot bisect each other.*

Proposition 61 *If two distinct circles intersect at two points, they cannot have the same center.*

Definition 62 *Two circles are said to be tangent if they "do not cut each other"; that is, if the angle between them (measured by the angle between tangents at the point of intersection) is zero.*

Proposition 63 *If two circles are tangent to one another, they cannot have the same center.*

Proposition 64 *Let P lie inside a circle and not be the center. Any chord through P is divided into a longer and shorter piece; the longest and shortest of these pieces are the pieces of the diameter through P and the pieces of a chord closer to the center are longer, resp. shorter, than the pieces of one further from the center.*

Proposition 65 *If lines are drawn from a point P outside a circle through to the concave piece of the circumference, then the one through the center is longest and those closer to the center are longer than those further from the center. But if lines are drawn from a point P outside a circle to the convex piece of the circumference, then the one that goes on through the center is shortest and those that go on closer to the center are shorter than those that go on further from the center. There are only two equal lines through P to the circle and they are on either side of the shortest.*

Proposition 66 *If P is within a circle and there are more than two chords through P of the same length, then P is the center of the circle.*

Proposition 67 *Two distinct circles meet at no more than two points.*

Proposition 68 *If two circles are tangent, one inside the other, then the line joining their centers passes through their point of tangency.*

Proposition 69 *If two circle are tangent, one outside the other, then the line joining their centers passes through their point of tangency.*

Proposition 70 *Two circles cannot be tangent at more than one point.*

Proposition 71 *Two chords have the same length if and only if they lie the same distance from the center.*

Proposition 72 *The longest chord in a circle is a diameter; and, given two chords, the chord closer to the center is longer.*

Proposition 73 *A perpendicular to a diameter at a point of intersection with the circle is a tangent to the circle.*

Proposition 74 *A tangent to a given circle through a given point outside the circle can be constructed.*

Proposition 75 *A tangent to a circle is perpendicular to a radius to the point of tangency.*

Proposition 76 *A line perpendicular to a tangent to a circle at the point of tangency will pass through the center of the circle.*

Proposition 77 *The measure of the central angle is double that of an inscribed angle if they intercept the same arc.*

Definition 78 *An angle on a chord $\overline{AB}$ is an inscribed angle in the major arc $\widehat{AB}$. (This means that the angle stands upon the minor arc.)*

Proposition 79 *Angles on the same chord are equal.*

Proposition 80 *The sum of opposite angles of a quadrilateral inscribed in a circle is 180°.*

Proposition 81 *A chord determines a unique arc on a circle.*

Proposition 82 *Chords in two circles with the same angle on them and of the same length determine congruent arcs.*

Proposition 83 *The circle can be constructed from an arc.*

Proposition 84 *In congruent circles equal angles stand on equal arcs if they are both central or both inscribed.*

Proposition 85 *In congruent circles angles standing on arcs of equal measure are equal if they are both central or both inscribed.*

Proposition 86 *Congruent chords of congruent circles cut off major and minor arcs of the same measure.*

Proposition 87 *Endpoints of arcs of the same measure on congruent circles determine chords of equal length.*

Proposition 88 *A given arc can be bisected.*

Proposition 89 *The angle inscribed in a semicircle is a right angle; that inscribed in a minor arc is greater than a right angle while that in a major arc is less than a right angle.*

Proposition 90 *The two supplementary angles made by a chord $\overline{AB}$ and a tangent to a circle at A are equal to the inscribed angles in the major and minor arcs determined by the chord respectively.*

Proposition 91 *Given an angle and a line segment, a circle and an arc on the circle can be constructed with the segment as a chord determining the arc such that the inscribed angle in the arc is congruent to the given angle.*

Proposition 92 *Given a circle and an angle, we can construct a chord cutting off an arc with an inscribed angle congruent to the given angle.*

Proposition 93 *If two chords cross each other in a circle, the two rectangles whose sides are the two segments of each chord have the same area.*

Proposition 94 *Choose a point P outside of a circle and two lines through P, one tangent to the circle at A and one cutting the circle in two points B and C. Then the area of the rectangle with sides of length equal to that of $\overline{PB}$ and $\overline{PC}$ is equal to the area of the square with sides of length $\overline{PA}$.*

Proposition 95 *Choose a point P outside of a circle and two lines through P, one cutting the circle in two points B and C and one meeting the circle at A. If the area of the rectangle with sides of length $\overline{PB}$ and $\overline{PC}$ is equal to the area of the square with side $\overline{PA}$, then $\overrightarrow{PA}$ is tangent to the circle*

Power of a point P with respect to a circle c : (Proposition 35 and 36) Let P be a point not on a circle and let two lines through P cut the circle at points A, B and C, D respectively. Then $\overline{PA} \cdot \overline{PB} = \overline{PC} \cdot \overline{PD}$.

Book IV

Proposition 96 *Given a circle and a line segment of length less than the diameter of the circle, a chord can be constructed congruent to the given segment.*

Proposition 97 *Given a circle and a triangle, a triangle can be inscribed in the circle with the same angles as the given triangle.*

Proposition 98 *Given a circle and a triangle, a triangle can be circumscribed around the circle with the same angles as the given triangle.*

Proposition 99 *A circle can be inscribed in a given triangle.*

Proposition 100 *A circle can be circumscribed around a given triangle.*

Proposition 101 *A square can be inscribed in a given circle.*

Proposition 102 *A square can be circumscribed around a given circle.*

Proposition 103 *A circle can be inscribed in a given square.*

Proposition 104 *A circle can be circumscribed around a given square.*

Proposition 105 *A* $36°$ - $72°$ - $72°$ *triangle can be constructed.*

Proposition 106 *An equilateral and equiangular pentagon can be inscribed in a given circle.*

Proposition 107 *An equilateral and equiangular pentagon can be circumscribed around a given circle.*

Proposition 108 *A circle can be inscribed in an equilateral and equiangular pentagon.*

Proposition 109 *A circle can be circumscribed around an equilateral and equiangular pentagon.*

Proposition 110 *A circle can be inscribed in an equilateral and equiangular hexagon.*

Proposition 111 *An equilateral and equiangular fifteen angled figure can be inscribed in a given circle.*

Book VI

Proposition 112 *Ratios of areas of triangles and parallelograms of the same height are the same as the ratio of their bases.*

Proposition 113 *A line drawn thru two sides of a triangle divides the sides into the same proportion if and only if it is parallel to the third side.*

Proposition 114 *In a triangle ABC, a line thru the vertex A divides $\overline{BC}$ into the same ratio as the sides $\overline{AB}$ and $\overline{AC}$ if and only if the line bisects the angle at A.*

Definition 115 *Two triangles ABC and $A'B'C'$ are similar if their angles are equal and the ratios of sides of corresponding angles are also equal.*

Proposition 116 *(AAA) Two triangles with the same angles are similar.*

Proposition 117 *Two similar triangles have the same angles opposite corresponding sides.*

Proposition 118 *(SAS) Two triangles with one angle equal and the ratios of the corresponding sides of the angle the same are similar.*

Proposition 119 *(AAS) If two triangles have one angle of each the same, the ratio of sides of a second angle the same, and either the third angles are both less than a right angle or are both greater than or equal to a right angle, then the triangles are similar.*

Proposition 120 *An altitude from the vertex of the right angle in a right triangle divides the right triangle into two right triangles, and all three of these right triangles are similar.*

Proposition 121 *A line segment can be divided into any given ratio.*

Proposition 122 *An arbitrary line segment can be divided in the same ratio as that of any given divided line segment.*

Proposition 123 *Double the ratio of two line segments can be constructed.*

Proposition 124 *Given the ratio of any two line segments, that proportion can be constructed using a third line segment.*

Proposition 125 *Given two line segments $\overline{AB}$ and $\overline{CD}$, a line segment $\overline{PQ}$ can be constructed so that $\frac{\overline{AB}}{\overline{PQ}} = \frac{\overline{PQ}}{\overline{CD}}$.*