9.1 – Verifying Trigonometric Identities and Using Trigonometric Identities to Simplify Trigonometric Expressions

Learning Objectives

1. Use Algebra to Simplify Trigonometric Expressions
2. Establish Identities

Identically Equal, Identity, and Conditional Equation

Two functions f and g are **identically equal** if

$$f(x) = g(x)$$

for every value of x for which both functions are defined. Such an equation is referred to as an **identity**. An equation that is not an identity is called a **conditional equation**.

For example, the following are identities:

$$(x+1)^2 = x^2 + 2x + 1 \qquad \sin^2 x + \cos^2 x = 1 \qquad \csc x = \frac{1}{\sin x}$$

The following are conditional equations:

$2x + 5 = 0$ — True only if $x = -\frac{5}{2}$

$\sin x = 0$ — True only if $x = k\pi$, k an integer

$\sin x = \cos x$ — True only if $x = \frac{\pi}{4} + 2k\pi$ or $x = \frac{5\pi}{4} + 2k\pi$, k an integer

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations.

In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations.

Summarizing Trigonometric Identities

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Even-Odd Identities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta$$
$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta \qquad \cot(-\theta) = -\cot\theta$$

1 – Use Algebra to Simplify Trigonometric Expressions

The ability to use algebra to manipulate trigonometric expressions is a key skill that one must have to establish identities. Four basic algebraic techniques are used to establish identities:

1. Rewriting a trigonometric expression in terms of sine and cosine only
2. Multiplying the numerator and denominator of a ratio by a "well-chosen 1"
3. Writing sums of trigonometric ratios as a single ratio
4. Factoring

Example 1 – Using Algebraic Techniques to Simplify Trigonometric Expressions

Simplify $\dfrac{\cot\theta}{\csc\theta}$ by rewriting each trigonometric function in terms of sine and cosine functions.

Solution:

Example 2 – Using Algebraic Techniques to Simplify Trigonometric Expressions

Show that $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$ by multiplying the numerator and denominatorby $1 - \sin\theta$.

Solution:

Example 3 – Using Algebraic Techniques to Simplify Trigonometric Expressions

Simplify $\dfrac{1 + \sin u}{\sin u} + \dfrac{\cot u - \cos u}{\cos u}$ by rewriting the expression as a single ratio.

Solution:

Example 4 – Using Algebraic Techniques to Simplify Trigonometric Expressions

Simplify $\dfrac{\sin^2 v - 1}{\tan v \sin v - \tan v}$ by factoring.

Solution:

Example 5 – Using Algebraic Techniques to Simplify Trigonometric Expressions

Simplify $2\cos^2\theta + \cos\theta - 1$ by factoring.
Solution:

In the examples that follow, the directions read "Establish the identity . . . ." This is accomplished by starting with one side of the given equation (usually the side containing the more complicated expression) and, using appropriate basic identities and algebraic manipulations, arriving at the other side. The selection of appropriate basic identities to obtain the desired result is learned only through experience and lots of practice.

Guidelines for Establishing Identities

- It is almost always preferable to start with the side containing the more complicated expression.
- Rewrite sums or differences of quotients as a single quotient.
- Sometimes it helps to rewrite one side in terms of sine and cosine functions only.
- Always keep the goal in mind. As you manipulate one side of the expression, keep in mind the form of the expression on the other side.

WARNING

Do not try to establish an identity by treating it as an equation. We cannot add or multiply both sides by the same expression because we do not know if the sides are equal. That is what we are trying to prove.

Example 6 – Establishing an Identity

Establish the identity: $\csc\theta \cdot \tan\theta = \sec\theta$

Solution:

Example 7 – Establishing an Identity

Establish the identity: $\sin^2(-\theta) + \cos^2(-\theta) = 1$

Solution:

Example 8 – Establishing an Identity

Establish the identity: $\dfrac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$

Solution:

Example 9 – Establishing an Identity

Establish the identity: $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\csc\theta$

Solution:

Example 10 – Establishing an Identity

Establish the identity: $$\frac{\tan v + \cot v}{\sec v \csc v} = 1$$

Solution: