8.3 – Inverse Trigonometric Functions

Learning Objectives

1. Understand and use the inverse sine, cosine, and tangent functions.
2. Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
3. Find exact values of composite functions with inverse trigonometric functions.

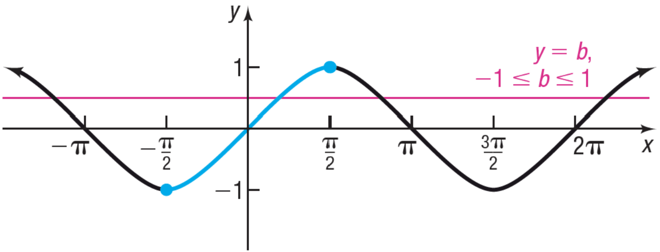
1 – Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In Sections 3.7 and 5.7 we discussed inverse functions, and we concluded that if a function is one-to-one, it will have an inverse function. We also observed that if a function is not one-to-one, it may be possible to **restrict** its domain in some suitable manner so that the restricted function is one-to-one.

For example, the function $f(x) = x^2$ is not one-to-one;however, if the domain is restricted to $[0, \infty)$, the new function is one-to-one.

We will use this same technique to define the inverse trigonometric functions.

Restricted Sine Function



From the HLT, the function $f(x) = \sin x$ is not one-to-one. We can restrict the domain of $f(x) = \sin x$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The restricted function

$$f(x) = \sin x \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

is one-to-one as has an inverse function.

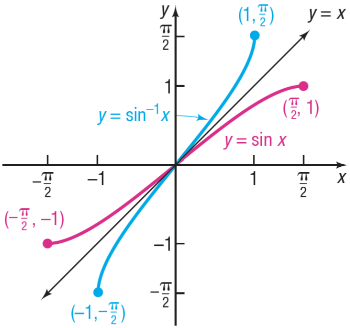
Inverse Sine Function

DEFINITION OF THE INVERSE SINE FUNCTION

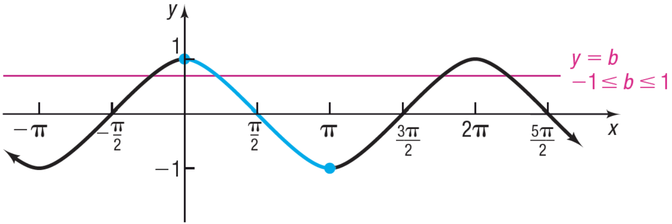
The **inverse sine function** is the function $\sin^{-1}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

The inverse sine function is also called **arcsine**, denoted by **arcsin**.



Restricted Cosine Function



From the HLT, the function $f(x) = \cos x$ is not one-to-one. We can restrict the domain of $f(x) = \cos x$ to $[0, \pi]$. The restricted function

$$f(x) = \cos x \quad 0 \leq x \leq \pi$$

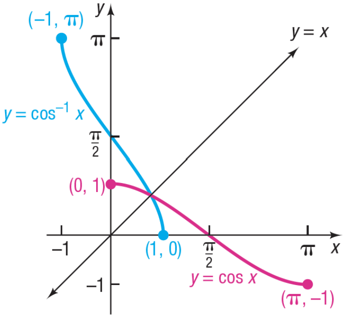
is one-to-one as has an inverse function.

DEFINITION OF THE INVERSE COSINE FUNCTION

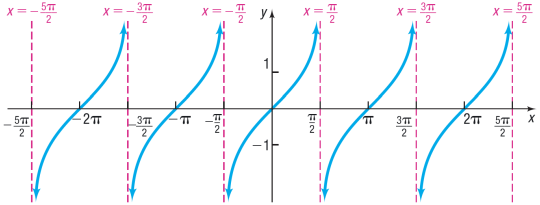
The **inverse cosine function** is the function $\cos^{-1}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

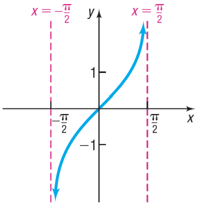
$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.



Restricted Tangent Function





From the HLT, the function $f(x) = \tan x$ is not one-to-one.

We can restrict the domain of $f(x) = \tan x$ to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The restricted function

$$f(x) = \tan x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

is one-to-one as has an inverse function.

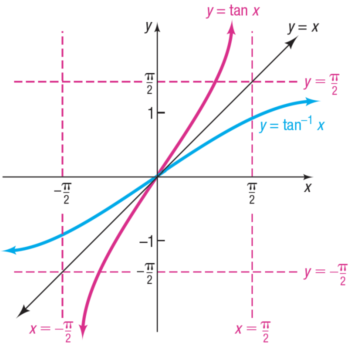
Inverse Tangent Function

DEFINITION OF THE INVERSE TANGENT FUNCTION

The **inverse tangent function** is the function $\tan^{-1}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.



Relations for Inverse Sine, Cosine, and Tangent Functions

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin\ y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos\ y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan\ y = x$, then $\tan^{-1} x = y$.

Example 1 – Writing a Relation for an Inverse Function

Given $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, write a relation involving the inverse sine.

Solution:

2 – Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles.

HOW TO

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Example 2 – Evaluating the Inverse Sine Function

Find each value.

(a) $\sin^{-1}\frac{1}{2}$ **(b)** $\sin^{-1}\left(-\frac{1}{2}\right)$ **(c)** $\sin^{-1}\frac{3}{2}$

Solution:

Example 3 – Evaluating the Inverse Cosine Function

Find each value.

(a) $\cos^{-1}\dfrac{\sqrt{3}}{2}$ **(b)** $\cos^{-1}0$ **(c)** $\cos^{-1}\left(-\dfrac{1}{2}\right)$

Solution:

Example 4 – Evaluating the Inverse Tangent Function

Find each value.

(a) $\tan^{-1} 1$ **(b)** $\tan^{-1}\sqrt{3}$ **(c)** $\tan^{-1}(20)$

Solution:

Example 5 – Evaluating Inverse Trigonometric Functions for Special Input Values

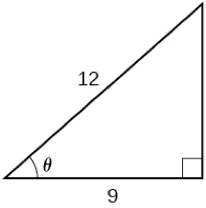
Evaluate each of the following.

a $\sin^{-1}\left(\frac{1}{2}\right)$ b $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ c $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ d $\tan^{-1}(1)$

Solution:

Example 6 – Applying the Inverse Cosine to a Right Triangle

Solve the triangle below for the angle θ.



Solution:

3 – Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output.

Compositions of a trigonometric function and its inverse

$$\sin(\sin^{-1} x) = x \text{ for } -1 \le x \le 1$$
$$\cos(\cos^{-1} x) = x \text{ for } -1 \le x \le 1$$
$$\tan(\tan^{-1} x) = x \text{ for } -\infty < x < \infty$$

$$\sin^{-1}(\sin\ x) = x \text{ only for } -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}$$
$$\cos^{-1}(\cos\ x) = x \text{ only for } 0 \le x \le \pi$$
$$\tan^{-1}(\tan\ x) = x \text{ only for } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}$$

Example 7 – Evaluating Expressions with Inverse Sine

Find each value.

(a) $\sin^{-1}\left(\sin \frac{\pi}{3}\right)$ **(b)** $\sin^{-1}\left(\sin \frac{2\pi}{3}\right)$

Solution:

Example 8 – Evaluating Expressions with Inverse Cosine

Find each value.

(a) $\cos^{-1}\left(\cos \dfrac{2\pi}{3}\right)$ **(b)** $\cos^{-1}\left(\cos \dfrac{5\pi}{3}\right)$

Solution:

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides.

Example 9 – Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution:

Example 10 – Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \leq x \leq 3$.

Solution: