8.1 and 8.2 – Graphs of Sine, Cosine and Tangent Functions

Learning Objectives

1. Graph $y = \sin x$ and $y = \cos x$.
2. Analyzing graphs of variations of $y = \sin x$ and $y = \cos x$.
3. Graph variations of $y = \sin x$ and $y = \cos x$.
4. Analyze the graph of $y = \tan x$.

We want to graph the sine and cosine functions in the $xy-$plane. So we use the traditional symbols x for the independent variable (or argument) and y for the dependent variable for each function. Then sine and cosine are written as

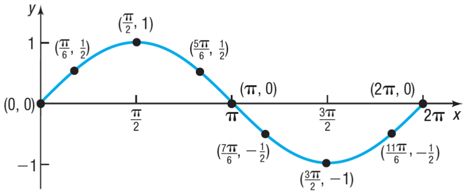
$$y = f(x) = \sin x \quad \text{and} \quad y = f(x) = \cos x$$

Here the independent variable x represents an angle, measured in radians.

Remember that the sine and cosine functions have a period of 2π. So it is only necessary to graph both functions on the interval $[0, 2\pi]$.

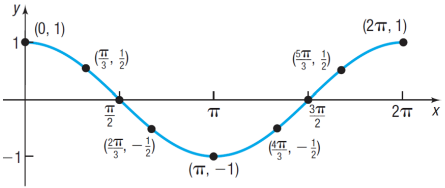
$y = \sin x,\ 0 \le x \le 2\pi$

x	$y = \sin x$	(x, y)
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{1}{2}$	$\left(\frac{\pi}{6}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	1	$\left(\frac{\pi}{2}, 1\right)$
$\frac{5\pi}{6}$	$\frac{1}{2}$	$\left(\frac{5\pi}{6}, \frac{1}{2}\right)$
π	0	$(\pi, 0)$
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{7\pi}{6}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	-1	$\left(\frac{3\pi}{2}, -1\right)$
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\left(\frac{11\pi}{6}, -\frac{1}{2}\right)$
2π	0	$(2\pi, 0)$



$y = \cos x,\ 0 \le x \le 2\pi$

x	$y = \cos x$	(x, y)
0	1	$(0, 1)$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\left(\frac{\pi}{3}, \frac{1}{2}\right)$
$\frac{\pi}{2}$	0	$\left(\frac{\pi}{2}, 0\right)$
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$
π	-1	$(\pi, -1)$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$\left(\frac{4\pi}{3}, -\frac{1}{2}\right)$
$\frac{3\pi}{2}$	0	$\left(\frac{3\pi}{2}, 0\right)$
$\frac{5\pi}{3}$	$\frac{1}{2}$	$\left(\frac{5\pi}{3}, \frac{1}{2}\right)$
2π	1	$(2\pi, 1)$



Characteristics of Sine and Cosine Functions

Characteristics of Sine and Cosine Functions

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $\left(-\infty, \infty\right)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about the y-axis, because it is an even function.

Sinusoidal Functions

A function that has the same general shape as a sine or cosine function is known as a sinusoidal function. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D$$

Determining the Period of Sinusoidal Functions

Period of Sinusoidal Functions

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx)$$
$$y = A\cos(Bx)$$

The period is $\frac{2\pi}{|B|}$.

Example 1 – Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\dfrac{\pi}{6}x\right)$.

Solution:

Determining Amplitude

In the general formula for a sinusoidal function, A represents the vertical stretch factor and its absolute value $|A|$ is called the **amplitude**.

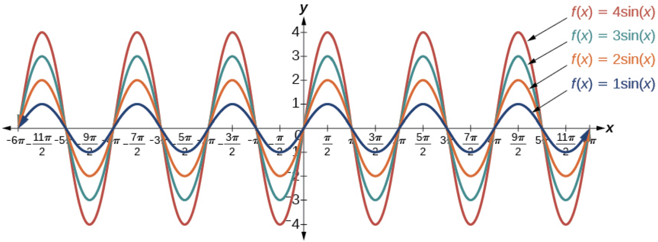
Amplitude of Sinusoidal Functions

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx)$$

The **amplitude** is $|A|$, which is the vertical height from the **midline** . In addition, notice in the example that

$$|A| = \text{amplitude} = \frac{1}{2}|\text{maximum} - \text{minimum}|$$



Example 2 – Identifying the Amplitude of a Sine or Cosine Function

Determine the amplitude and period of the sinusoidal function $f(x) = 3\sin(4x)$.

Solution:

2 – Analyzing Graphs of Variations of $y = \sin x$ and $y = \cos x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D.

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

$$\textit{or}$$

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \text{ and } y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D$$

The value $\dfrac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left.

D indicates the vertical shift from the midline in the general formula for a sinusoidal function. Any value of D other than zero shifts the graph up or down.

Variations of Sine and Cosine Functions

Variations of Sine and Cosine Functions

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the vertical shift.

Example 3 – Identifying the Phase and Vertical Shifts of a Function

Determine the phase shift and the vertical shift for
$f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2.$
Solution:

HOW TO

Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$, identify the midline, amplitude, period, and phase shift.

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

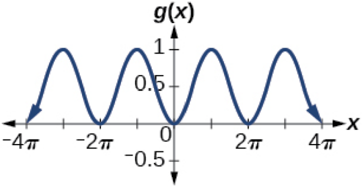
Example 4 – Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution:

Example 5 – Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function



Solution:

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Graph Sinusoidal Functions Using Key Points

Steps for Graphing a Sinusoidal Function of the Form $y = A\sin(Bx)$ or $y = A\cos(Bx)$ Using Key Points

STEP 1: Determine the amplitude and period of the sinusoidal function.

STEP 2: Divide the interval $\left[0, \dfrac{2\pi}{B}\right]$ into four subintervals of the same length.

STEP 3: Use the endpoints of these subintervals to obtain five key points on the graph.

STEP 4: Plot the five key points, and draw a sinusoidal graph to obtain the graph of one cycle. Extend the graph in each direction to make it complete.

Example 6 – Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution:

Example 7 – Graphing a Sinusoidal Function Using Key Points

Graph $y = 2\sin\left(-\frac{\pi}{2}x\right)$ using key points.

Solution:

Example 8 – Graphing a Sinusoidal Function Using Key Points

Graph $y = -4\cos(\pi x) - 2$ using key points. Use the graph to determine the domain and the range of $y = -4\cos(\pi x) - 2$.

Solution:

Example 9 – Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Solution:

Example 10 – Identifying the Properties of a Sinusoidal Function

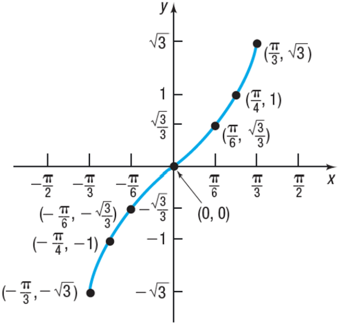
Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and vertical shift. Then graph the function.

Solution:

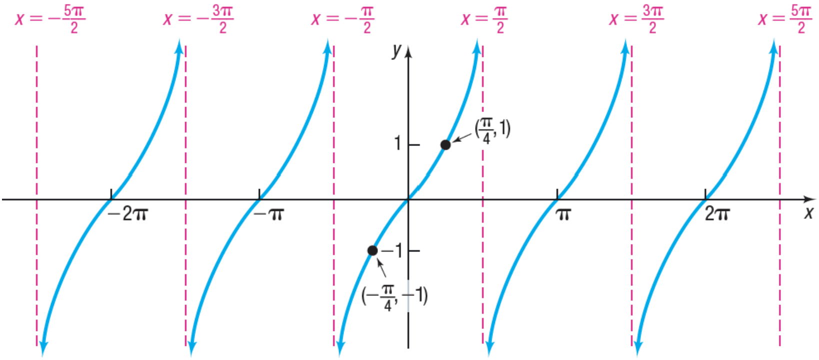
Properties of the Tangent Function

- The domain is the set of all real numbers, except odd multiples of $\frac{\pi}{2}$.
- The range is the set of all real numbers.
- The tangent function is an odd function, as the symmetry of the graph with respect to the origin indicates.
- The tangent function is periodic, with period π.
- The x-intercepts are $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$; the y-intercept is 0.
- Vertical asymptotes occur at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

x	$y = \tan x$	(x, y)
$-\frac{\pi}{3}$	$-\sqrt{3} \approx -1.73$	$\left(-\frac{\pi}{3}, -\sqrt{3}\right)$
$-\frac{\pi}{4}$	-1	$\left(-\frac{\pi}{4}, -1\right)$
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3} \approx -0.58$	$\left(-\frac{\pi}{6}, -\frac{\sqrt{3}}{3}\right)$
0	0	$(0, 0)$
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3} \approx 0.58$	$\left(\frac{\pi}{6}, \frac{\sqrt{3}}{3}\right)$
$\frac{\pi}{4}$	1	$\left(\frac{\pi}{4}, 1\right)$
$\frac{\pi}{3}$	$\sqrt{3} \approx 1.73$	$\left(\frac{\pi}{3}, \sqrt{3}\right)$



$y = \tan x, -\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$



$x = -\frac{5\pi}{2}$
$x = -\frac{3\pi}{2}$
$x = -\frac{\pi}{2}$
$x = \frac{\pi}{2}$
$x = \frac{3\pi}{2}$
$x = \frac{5\pi}{2}$
y
1
$(\frac{\pi}{4}, 1)$
-2π
$-\pi$
π
2π
x
-1
$(-\frac{\pi}{4}, -1)$