7.4 – The Other Trigonometric Functions

Learning Objectives

1. Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.
2. Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
3. Use properties of even and odd trigonometric functions.
4. Recognize and use fundamental identities.

Tangent, Secant, Cosecant, and Cotangent Functions

If t is a real number and (x, y) is a point where the terminal side of an angle t of radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0$$

$$\sec t = \frac{1}{x}, \neq 0$$

$$\csc t = \frac{1}{y}, y \neq 0$$

$$\cot t = \frac{x}{y}, y \neq 0$$

Example 1 – Finding Trigonometric Functions from a Point on the Unit Circle

The point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is on the unit circle. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

Solution:

Example 2 – Finding the Trigonometric Functions of an Angle

Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$, when $t = \dfrac{\pi}{6}$.

Solution:

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

2 – Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

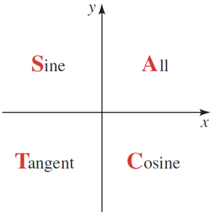
We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions.The procedure is the same: Find the reference angle formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by $x-$ and $y-$values in the original quadrant.

HOW TO

Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions.

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

The following mnemonic device will help you remember which trigonometric functions are positive in each quadrant:



SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

You can remember this as "All Students Take Calculus."

Example 3 – Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\dfrac{5\pi}{6}$.

Solution:

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input.

Recall from section 3.5 – Transformations of Functions, we defined what it means for a function to be **even** and what it means for a function to be **odd**.

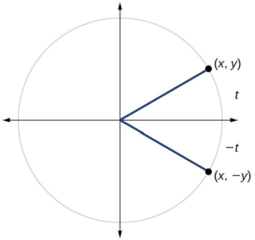
Even and Odd Functions

A function is called an **even function** if for every input x $f(x) = f(-x)$.

A function is called an **odd function** if for every input x $f(x) = -f(x)$.

We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle.

The sine of the positive angle is y. The sine of the negative angle is $-y$. The sine function, then, is an odd function. We can test each of the six trigonometric functions in this fashion.



$\sin t = y$ $\sin(-t) = -y$ $\sin t \neq \sin(-t)$	$\cos t = x$ $\cos(-t) = x$ $\cos t = \cos(-t)$	$\tan(t) = \frac{y}{x}$ $\tan(-t) = -\frac{y}{x}$ $\tan t \neq \tan(-t)$
$\sec t = \frac{1}{x}$ $\sec(-t) = \frac{1}{x}$ $\sec t = \sec(-t)$	$\csc t = \frac{1}{y}$ $\csc(-t) = \frac{1}{-y}$ $\csc t \neq \csc(-t)$	$\cot t = \frac{x}{y}$ $\cot(-t) = \frac{x}{-y}$ $\cot t \neq \cot(-t)$

Even and Odd Trigonometric Functions

Cosine and secant are even:

$$\cos(-t) = \cos t$$

$$\sec(-t) = \sec t$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t$$

$$\tan(-t) = -\tan t$$

$$\csc(-t) = -\csc t$$

$$\cot(-t) = -\cot t$$

Example 4 – Using Even and Odd Properties of Trigonometric Functions

If $\sec t = 2$, find $\sec(-t)$.
Solution:

Example 5 – Finding Exact Values Using Even–Odd Properties

Find the exact value of each of the following:

(a) $\sin(-45°)$ (b) $\cos(-\pi)$ (c) $\cot(-120°)$ (d) $\tan\left(-\dfrac{37\pi}{4}\right)$

Solution:

4 – Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further,and derive some fundamental identities.

Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know.

For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

Fundamental Identities

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t}$$

$$\sec t = \frac{1}{\cos t}$$

$$\csc t = \frac{1}{\sin t}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$$

Example 6 – Using Identities to Evaluate Trigonometric Functions

(a) Given $\sin(45°) = \frac{\sqrt{2}}{2}, \cos(45°) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45°)$.

(b) Given $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}, \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, evaluate $\sec\left(\frac{5\pi}{6}\right)$.

Solution:

Example 7 – Using Identities to Simplify Trigonometric Expressions

Simplify $\dfrac{\sec t}{\tan t}$.

Solution:

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternate forms of the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$1 + \tan^2 t = \sec^2 t$$

The other form is obtained by dividing both sides by $\sin^2 t$ to get $\cot^2 t + 1 = \csc^2 t$.

Alternate Forms of the Pythagorean Identity

$$1 + \tan^2 t = \sec^2 t$$

$$\cot^2 t + 1 = \csc^2 t$$

Example 8 – Using Identities to Relate Trigonometric Functions

If $\cos t = \dfrac{12}{13}$ and t is in quadrant IV, find the values of the other five trigonometric functions.

Solution:

Periodic Functions

A period is the shortest interval over which a function completes one full cycle.

For example, the lengths of months repeat every four years. If x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4) = f(x)$.

This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The period is 4

The trigonometric functions are periodic.

Period of a Function

The **period** P of a repeating function is the number representing the interval such that $f(x+P) = f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.
The period of the tangent and cotangent functions is π.

Example 9 – Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on