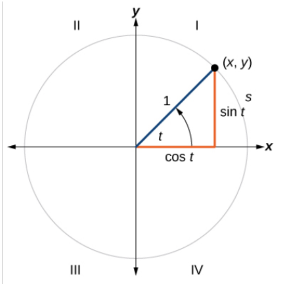
7.3 – Unit Circle

Learning Objectives

1. Find function values for the sine and cosine 30° or $\left(\dfrac{\pi}{6}\right)$, 45° or $\left(\dfrac{\pi}{4}\right)$, and 60° or $\left(\dfrac{\pi}{3}\right)$.
2. Identify the domain and range of sine and cosine functions.
3. Find reference angles.
4. Use reference angles to evaluate trigonometric functions.

1 – Finding Trigonometric Functions Using the Unit Circle



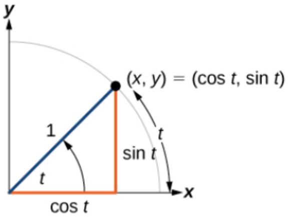
Unit circle where the central angle is *t* radians

A **unit circle** has a has a center at $(0, 0)$ and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.
Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions from the Unit Circle

The **sine function** of an angle t equals the $y-$value of the endpoint on the unit circle of an arc of length t.

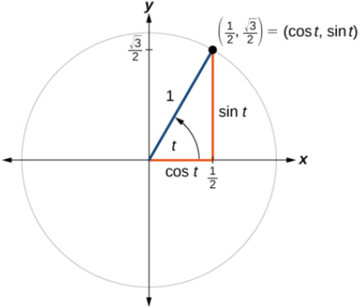
The **cossine function** of an angle t equals the $x-$value of the endpoint on the unit circle of an arc of length t.



Sine and Cosine Functions
If t is a real number and a point (x, y) on the unit circle corresponds to a central angle t, then $\cos t = x$ and $\sin t = y$.

Example 1 – Finding Function Values for Sine and Cosine

The point P is on the unit circle corresponding to an angle t, as shown below. Find $\cos t$ and $\sin t$.



Solution:

Example 2 – Finding Function Values for Sine and Cosine

A certain angle t corresponds to a point on the unit circle at $\left(-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$. Find $\cos t$ and $\sin t$.

Solution:

Finding Sines and Cosines of Angles on an Axis

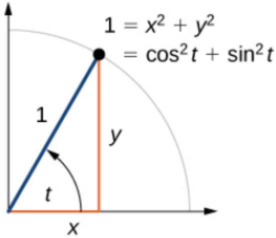
For **quadrantral angles**, the corresponding point on the unit circle falls on the $x-$ or $y-$axis. In that case, we can easily calculate cosine and sine from the values of x and y

Example 3 – Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.
Solution:

The Pythagorean Identity

The **Pythagorean Identity** states that, for any real number t, $\cos^2 t + \sin^2 t = 1$.



HOW TO

Given the sine of some angle t and its quadrant location, find the cosine of t.

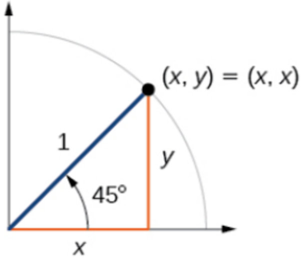
1. Substitute the known value of $\sin t$ into the Pythagorean Identity.
2. Solve for $\cos t$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

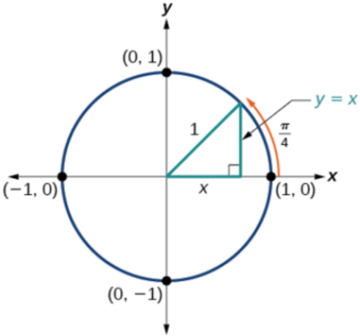
Example 4 – Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin t = \dfrac{3}{7}$ and t is in the second quadrant, find $\cos t$.

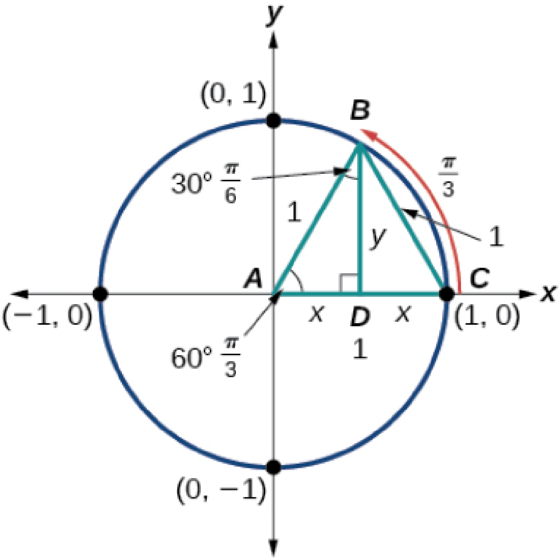
Solution:

Finding Sines and Cosines of 45° Angles





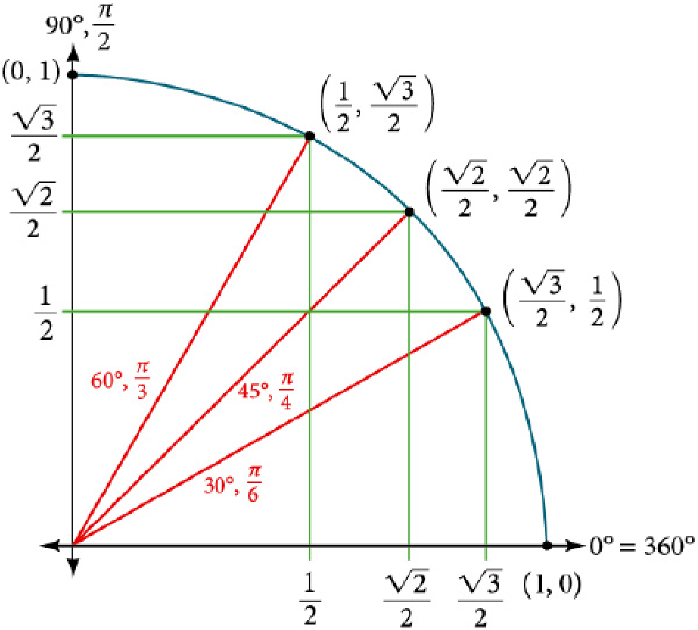
Finding Sines and Cosines of 30° and 60° Angles



Finding Sines and Cosines of Special Angles

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. The table summarizes these values.

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1



$90°, \frac{\pi}{2}$
$(0, 1)$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\sqrt{3}}{2}$
$\frac{\sqrt{2}}{2}$
$\frac{1}{2}$
$60°, \frac{\pi}{3}$
$45°, \frac{\pi}{4}$
$30°, \frac{\pi}{6}$
$0° = 360°$
$\frac{1}{2}$
$\frac{\sqrt{2}}{2}$
$\frac{\sqrt{3}}{2}$
$(1, 0)$

2 – Identifying the Domain and Range of Sine and Cosine Functions

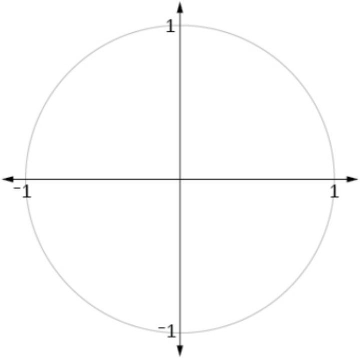
Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges.

What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions?

Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y and r there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions.
The input to the sine and cosine functions is the rotation from the positive $x-$axis. So the domain of sine and cosine functions is $(-\infty, \infty)$.

Range of Sine and Cosine

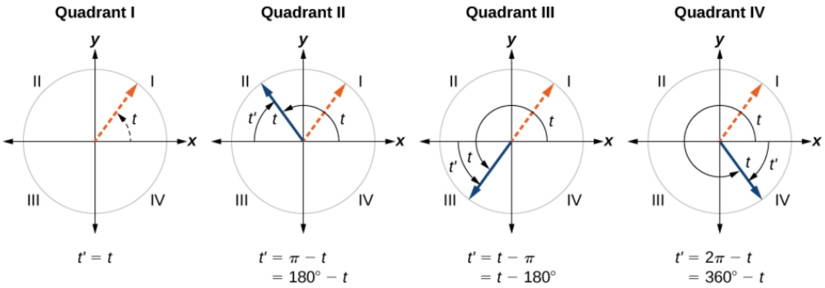
What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output?



The bounds of the $x-$coordinate are $[-1, 1]$. The bounds of the $y-$coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

Reference Angle

Let t be an angle in standard position. The reference angle t' associated with t is the acute angle formed by the terminal side of t and the $x-$axis.



Example 5 – Finding a Reference Angle

Find the reference angle for each of the following angles:

(a) $150°$ (b) $-45°$ (c) $\dfrac{9\pi}{4}$ (d) $-\dfrac{5\pi}{6}$

Solution:

4 – Using Reference Angles to Evaluate Trigonometric Functions

Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

HOW TO

Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

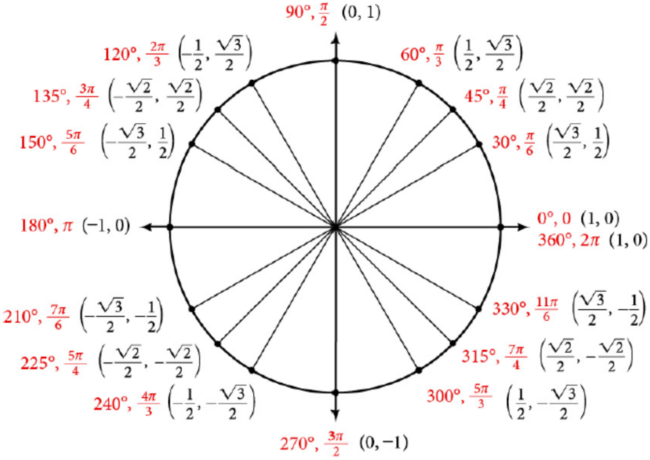
Example 6 – Using Reference Angles to Find Sine and Cosine

(a) Using a reference angle, find the exact value of $\cos(150°)$ and $\sin(150°)$.

(b) Using the reference angle, find $\cos \frac{5\pi}{4}$ and $\sin \frac{5\pi}{4}$.

Solution:

Special angles and coordinates of corresponding points on the unit circle



Using Reference Angles to Find Coordinates

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the identities $x = \cos t$ and $y = \sin t$.
First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the reference angle, and give them the signs corresponding to the $y-$and $x-$values of the quadrant.

HOW TO

Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 7 – Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\dfrac{7\pi}{6}$.

Solution: