7.1 – Angles

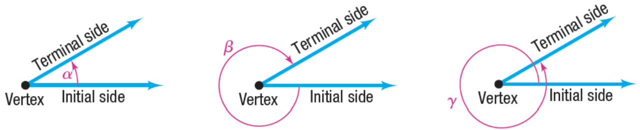
Learning Objectives

1. Draw angles in standard position.
2. Convert between degrees and radians.
3. Find coterminal angles.
4. Find the length of a circular arc.

A **ray** is a portion of a line that starts at a point V on the line and extends indefinitely in one direction. The starting point V of a ray is called its **vertex**.

When two rays are drawn with a common vertex, they form an **angle**. We call one ray of an angle the **initial side** and the other the **terminal side**. The angle formed is identified by showing the direction and amount of rotation from the initial side to the terminal side. If the rotation is in the counterclockwise direction, the angle is **positive**; if the rotation is clockwise, the angle is **negative**.

Angles



(a) Counterclockwise rotation Positive angle

(b) Clockwise rotation Negative angle

(c) Counterclockwise rotation Positive angle

Lowercase Greek letters, such as α (alpha), β (beta), γ (gamma),ϕ (phi) and θ (theta), are often used to denote angles.

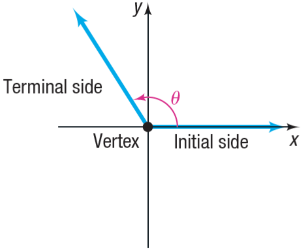
θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

Angle Measure and Angles in Standard Position

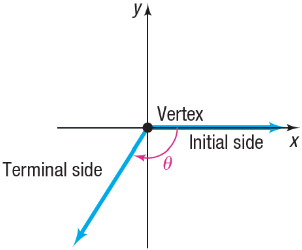
The **measure of an angle** is the amount of rotation from the initial side to the terminal side.

Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation.

An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive x−axis.



(a) θ is in standard position; θ is positive

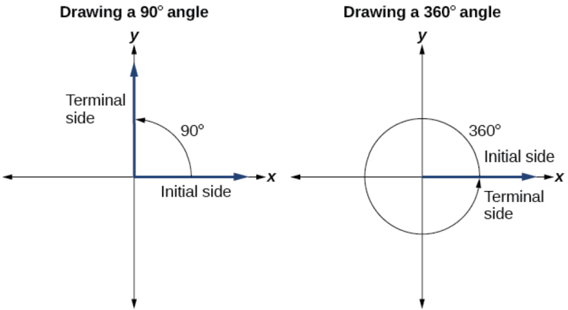


(b) θ is in standard position; θ is negative

Drawing an Angle in Standard Position

Drawing an angle in standard position always starts the same way—draw the initial side along the positive $x-$axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents.

For example, to draw a 90° angle, we calculate that $\dfrac{90^\circ}{360^\circ} = \dfrac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive x-axis.



When an angle θ is in standard position, either the terminal side will lie in a quadrant, in which case we say that θ **lies in that quadrant**, or the terminal side will lie on the $x-$axis or the $y-$axis, in which case we say that θ is a **quadrantal angle**.

Quadrantal Angles

An angle is a **quadrantal angle** if its terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

HOW TO

Given an angle measure in degrees, draw the angle in standard position.

1. Express the angle measure as a fraction of $360°$.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 1 – Drawing an Angle in Standard Position Measured in Degrees

(a) Sketch an angle of $30°$ in standard position.

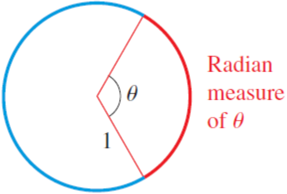
(b) Sketch an angle of $-135°$ in standard position.

Solution:

In calculus and other branches of mathematics a more natural method of measuring angles is used: **radian measure**.

DEFINITION OF RADIAN MEASURE

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle.



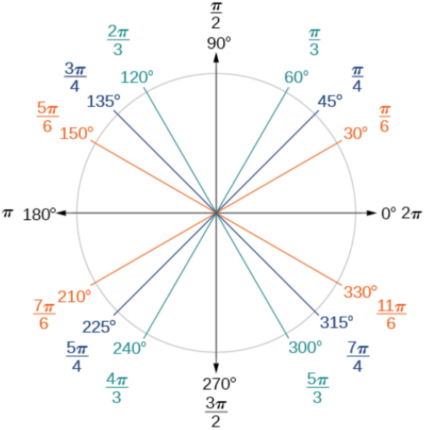
Using Radians

Consider the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned}
1 \text{ rotation} &= 360^\circ &= 2\pi \text{ radians} \\
\tfrac{1}{2} \text{ rotation} &= 180^\circ &= \pi \text{ radians} \\
\tfrac{1}{4} \text{ rotation} &= 90^\circ &= \tfrac{\pi}{2} \text{ radians}
\end{aligned}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees.



Example 2 – Finding a Radian Measure

Find the radian measure of one-third of a full rotation.
Solution:

Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them.

RELATIONSHIP BETWEEN DEGREES AND RADIANS

$$180^\circ = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \qquad 1^\circ = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\dfrac{180}{\pi}$.

Example 3 – Converting Radians to Degrees

Convert each angle in radians to degrees.

(a) $\frac{\pi}{6}$ radian (b) $\frac{3\pi}{2}$ radians (c) $-\frac{7\pi}{4}$ radians (d) $\frac{7\pi}{3}$ radians (e) 3 radians

Solution:

Example 4 – Converting from Degrees to Radians

Convert each angle in degrees to radians.

(a) 60° (b) 150° (c) -45° (d) 90° (e) 107°

Solution:

3 – Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

Coterminal Angles

Coterminal angles are two angles in standard position that have the same terminal side.

Any angle has infinitely many coterminal angles because each time we add $360^\circ/2\pi$ or subtract $360^\circ/2\pi$ from it – the resulting value has a terminal side in the same location.

Example 5 – Finding Coterminal Angles

(a) Find angles that are coterminal with the angle $\theta = 30°$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \dfrac{\pi}{3}$ in standard position.

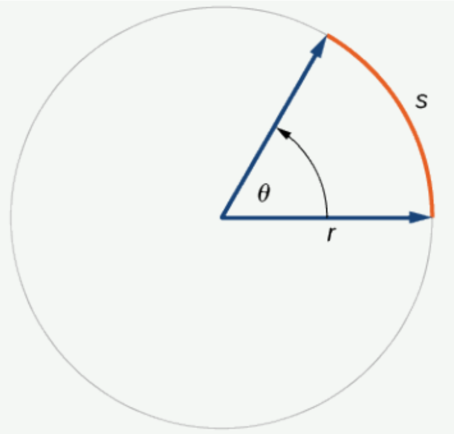
Solution:

Example 6 – Finding Coterminal Angles

Find an angle with measure between $0°$ and $360°$ that is coterminal with the angle of measure $1290°$ in standard position.

Solution:

In a circle of radius r, the length of an arc subtended by an angle with measure in radians is $s = r\theta$.



Example 7 – Finding the Length of an Arc

(a) Find the length of an arc of a circle with radius 10 m that subtends a angle of $30°$.

(b) An angle θ in a circle of radius 4 m is subtended by an arc of length 6 m. Find the measure of θ in radians.

Solution: