6.7 – Exponential Models

Learning Objectives

1. Define exponential growth and decay
2. Investigate continuous growth
3. Model exponential growth and decay

What exactly does it mean to grow exponentially? What does the word double have in common with percent increase?

1. **Percent change** refers to a change based on a percent of the original amount.
2. **Exponential growth** refers to an increase based on a constant multiplicative rate of change over equal increments of time, that is, a percent increase of the original amount over time.
3. **Exponential decay** refers to a decrease based on a constant multiplicative rate of change over equal increments of time, that is, a percent decrease of the original amount over time.

Exponential Growth vs Linear Growth

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

1. **Exponential growth** refers to the original value from the range increases by the same percentage over equal increments found in the domain.
2. **Linear growth** refers to the original value from the range increases by the same amount over equal increments found in the domain.

The general form of the exponential function is $f(x) = ab^x$, where a is any nonzero number, b is a positive number $\neq 1$.

1. If $b > 1$ the function grows at a rate proportional to its size – Exponential Growth
2. If $0 < b < 1$, the function decays at a rate proportional to its size – Exponential Decay

Exponential Growth

A function that models **exponential growth** grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$$f(x) = ab^x$$

where

1. a is the initial or starting value of the function.
2. b is the growth factor or growth multiplier per unit x.

Example 1 – Evaluating a Real-World Exponential Model

The population of India was about 1.25 billion in the year 2013 with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution:

Example 2 – Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an exponential function $N(t)$ representing the population (N) of deer over time.

Solution:

2 – Investigating Continuous Growth

Exponential models that use e as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

The Continuous Growth/Decay Formula

For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t) = ae^{rt}$$

where

1. a is the initial value
2. r is the continuous growth rate per unit time
3. and t is the elapsed time

If $r > 0$, then the formula represents continuous growth.

If $r < 0$, then the formula represents continuous decay.

Example 3 – Calculating Continuous Decay

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?
Solution:

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function.

We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model.

We may use the exponential decay model when we are calculating **half-life**, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

Exponential Growth (Doubling Time)

If the initial size of a population is n_0 and the doubling time is a, then the size of the population at time t is

$$N(t) = n_0 2^{t/a}$$

where a and t are measured in the same time units (minutes, hours, days, years, and so on).

Example 4 – Bacteria Population

Under ideal conditions a certain bacteria population doubles every three hours. Initially, there are 1000 bacteria in a colony.

(a) Find a model for the bacteria population after t hours.

(b) How many bacteria are in the colony after 15 hours?

(c) After how many hours will the bacteria count reach 100,000?

Solution:

Example 5 – Rabbit Population

A certain breed of rabbit was introduced onto a small island 8 months ago. The current rabbit population on the island is estimated to be 4100 and doubling every 3 months.

(a) What was the initial size of the rabbit population?

(b) Estimate the population 1 year after the rabbits were introduced to the island.

Solution:

Exponential Growth (Continuous Growth)

A population that experiences **exponential growth** increases according to the model

$$A(t) = n_0 e^{rt}$$

where

1. $A(t)$ = population at time t
2. n_0 = initial size of the population
3. r = continuous growth rate (expressed as a proportion of the population)
4. t = time

Example 6 – Predicting the Size of a Population

The initial bacterium count in a culture is 500. A biologist later makes a sample count of bacteria in the culture and finds that the continuous growth rate is 40% per hour.

(a) Find a function that models the number of bacteria after t hours.

(b) What is the estimated count after 10 hours?

(c) After how many hours will the bacteria count reach 80,000?

Solution:

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. The rate of decay is proportional to the mass of the substance.

This is analogous to population growth except that the mass decreases.

Physicists express the rate of decay in terms of **half-life**, the time it takes for a sample of the substance to decay to half its original mass.

In general, for a radioactive substance with mass m_0 and half-life h, the amount remaining at time t is modeled by

$$m(t) = m_0 2^{-t/h}$$

where h and t are measured in the same time units (minutes, hours, days, years, and so on).

Continuous Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$ is the **continuous decay rate**.

Example 7 – Radioactive Decay

Polonium-210 (^{210}Po) has a half-life of 140 days. Suppose a sample of this substance has a mass of 300 mg.

(a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t days.

(b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t days.

(c) Find the mass remaining after one year.

(d) How long will it take for the sample to decay to a mass of 200 mg?

Solution: